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editor: Leopoldo Nachbin

Functional Analysis: Surveys and Recent Results III

KLAUS-DIETER BIERSTEDT
BENNO FUCHSSTEINER
editors

NORTH-HOLLAND

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Editor: Leopoldo Nachbin

Centro Brasileiro de Pesquisas Fisicas
and University of Rochester

Functional Analysis: Surveys and Recent Results III

Proceedings of the Conference on Functional Analysis
Paderborn, Germany, 24-29 May, 1983

Edited by
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and
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PREFACE

The First and Second Conferences on Functional Analysis at the University of Paderborn had been held in November 1976 and February 1979; their Proceedings were published (in 1977 and 1980) as volumes 27 (Notas de Matemática 63) and 38 (Notas de Matemática 68) of the North-Holland Mathematics Studies series, respectively.

The Third Paderborn Conference on Functional Analysis took place from May 24 to May 29, 1983 (i. e., it was one day longer than the previous meetings). Organizers and editors of this volume, which contains the Proceedings of the conference, were again K.-D. Bierstedt and B. Fuchssteiner . - Many of the invited lecturers of the first two meetings also attended the 1983 conference; some of them gave short lectures, and others served as chairmen of the sessions.

At the 1983 Paderborn Conference, there were 17 invited lectures (of 50 or 75 minutes each) on topics of current active research in functional analysis, operator theory, and related areas. Again, in accordance with the aim of this series of meetings, most of the speakers of the invited lectures first presented a survey of the theory, some motivation and background information in the first part of their talk before proceeding to recent contributions and new results.

Since this conference was one day longer than those of 1977 and 1980, it was also possible to include 8 short lectures (of 25 minutes each), and four articles in this book were contributed from this part of the participants. - The joint paper of H. Goldmann, B. Kramm and D. Vogt is based on discussions during the meeting, and for its origins see the remark at the end of (3.1) in Kramm's article. Furthermore, one article was submitted by L. A. Moraes, a participant of the conference.

As a glance at the table of contents shows, the 22 articles in this volume deal with many different aspects of functional analysis and its applications, ranging from Banach space theory and the theory of (nuclear) Fréchet spaces to e.g. positive operators, automatic continuity, C* - and von Neumann algebras, function algebras, distribution theory and convolution operators.

At the time when the manuscripts for this Proceedings volume kept arriving at Paderborn, we received the sad news of B. Kramm's tragic death on October 11, 1983. All the participants of the conference will remember Bruno Kramm as an active and very energetic mathematician, full of interesting ideas. We all deeply regret this great loss.

The original plans for Kramm's survey article in these Proceedings included 8 chapters of which, due to his severe illness this summer and fall, only 4 could actually be finished. In the preparation of his contribution (and of the manuscript in this volume), Kramm was helped by his student, H. Goldmann. In fact, as time went on and Kramm's health situation became critical, Goldmann was, more and more, on his own, turning the notes into the final version of the paper.

Of course, the present article "Nuclearity and function algebras - a survey -" by Bruno Kramm is not really what Kramm had in mind when he started writing it and what the editors had hoped to receive. E.g., it does not contain an introduction and breaks off at the end of chapter 4. However, in its organization and in the inclusion of problems and interesting remarks, it clearly reflects Kramm's spirit and some of his research on the borderlines between functional analysis, several complex variables and function algebras. So we decided to include the manuscript in these Proceedings.

It remains to thank once more all those who participated in the meeting for their interest and the stimulating discussions, above all the speakers and the chairmen of the sessions. We thank all the contributors for the preparation of their manuscripts in time for the publication (the deadline was, more or less, September 30, 1983, but the last paper actually arrived at Paderborn in late November). We thank the analysis group of Paderborn for their help in organizing the meeting and in proofreading the manuscripts; we mention J. Cioranescu, B. Ernst, R. Hollstein and, above all, Wolfgang Lusky in this connection. Ms. B. Duddeck did an excellent job with the typing of two manuscripts (and the first pages of the book) as well as with the correction of many misprints.

Finally, we would like to thank "Stiftung Volkswagenwerk" for providing the funds for the conference, Universität-Gesamthochschule Paderborn for all the support, the editor of Notas de Matemática, Leopoldo Nachbin, and the publisher for including this volume in their series.

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SCHEDULE OF LECTURES

Wednesday, May 25, 1983:

Morning Session

9.00 Opening Session

Chairman: J. Schmets (Liège)

9.30 - 10.45 H. G. Dales (Leeds), Automatic continuity of homomorphisms from C*-algebras

Chairman: S. D. Chatterji (Lausanne)

11.15 - 12.30 E. G. F. Thomas (Groningen), The theorem of Bochner-Schwartz-Godement for generalized Gelfand pairs

Afternoon Session

Chairman: G. Maltese (Münster)

2.30 - 3.20 H. H. Schaefer (Tübingen), Positive bilinear forms and the Radon-Nikodym theorem

3.30 - 4.20 R. Nagel (Tübingen), What can positivity do for stability?

Chairman: E. Albrecht (Saarbrücken)

4.40 - 5.55 K. B. Laursen (Copenhagen), Commutative homomorphisms from C*-algebras

Thursday, May 26, 1983:

Morning Session

Chairman: T. Terzioglu (Ankara /Wuppertal)

9.00 - 10.15 D. Vogt (Wuppertal), Recent results on continuous linear maps between Frêchet spaces

Chairman: W. Kaballo (Dortmund)

10.25 - 11.15 R. Meise (Düsseldorf), Power series spaces of infinite type in discrete weighted interpolation theory and as kernels of convolution operators

12.05 - 12.30 V. Aurich (München), Bounded holomorphic embeddings of the unit disk into Banach spaces

Afternoon Session

Chairman: H. G. Tillmann (Münster)

2.30 - 3.45 H. P. Lotz (Urbana /Tübingen), Uniform convergence of operators in L_∞ and similar spaces

3.55 - 4.20 R. M. Aron (Dublin), Infinite dimensional polydisc algebras

Chairman: M. Neumann (Essen)

4.40 - 5.30 L. Weis (Kaiserslautern), Decompositions of positive operators and some of their applications

5.40 - 6.05 P. Pêrez Carreras (Valencia), Preservation of barreledness properties in tensor products

Friday, May 27, 1983:

Morning Session

Chairman: H. König (Kiel)

9.00 - 10.15 P. Wojtaszczyk (Warszawa), The Banach space H_1

Chairman: B. Gramsch (Mainz)

10.45 - 12.00 A. Pełczyński (Warszawa), Critical exponents

12.10 - 12.35 C. Schütt (Kiel), On the positive projection constant

Afternoon Session

Chairman: H. J. Petzsche (Düsseldorf)

2.30 - 3.45 B. Kramm (Bayreuth), Nuclearity and function algebras

Chairman: M. Wolff (Tübingen)

4.05 - 4.55 K. Donner (München), Extension of operators and approximation by linear operators

5.05 - 5.30 H. v. Weizsäcker (Kaiserslautern), Orthogonal kernels

Saturday, May 28, 1983:

Morning Session

Chairman: G. Hegerfeldt (Göttingen)

9.00 - 9.50 G. Wittstock (Saarbrücken), Matrix convex analysis on operator algebras

10.00 - 10.25 D. Petz (Budapest), Ergodic theorems in von Neumann algebras

Chairman: K. Floret (Kiel)

10.55 - 12.10 P. Dierolf (Trier), Multiplication operators between spaces of distributions

12.20 - 12.45 R. Mennicken (Regensburg), An elementary proof of a theorem of Keldyš

Afternoon Session

Chairman: A. Costê (Dakar/Caen)

2.45 - 3.35 H. Jarchow (Zürich), Hilbertisable topologies on Banach spaces

Chairman: W. Lusky (Paderborn)

4.00 - 4.50 C. Stegall (Linz), A class of topological spaces and differentiation of functions on Banach spaces

5.00 - 5.25 W. Ruess (Essen), Geometry of operator spaces.

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Functional Analysis: Surveys and Recent Results III
K.-D. Bierstedt and B. Fuchssteiner (eds.)

THE BANACH SPACE H_1

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We present the linear - topological properties of the classical Hardy space H_1.

INTRODUCTION

The aim of this paper is to give an exposition of linear - topological and isometric properties of the classical Hardy space $H_1(D)$. The importance of the space $H_1(D)$ and its versions like $ReH_1(T)$ in analysis stems from the fact that many integral operators which are unbounded on L_1 are bounded on H_1. We will not elaborate on this point here. While writing this paper we had two goals in mind. First, we wanted to show to a functional analyst an interesting and natural example of Banach space. In our opinion the Banach space properties of $H_1(D)$ are not well understood. There are many interesting and difficult problems. Secondly to a classical analyst the functional analytic point of view may be the source of new problems.

Our presentation of the subject is limited to $H_1(D)$. The fact that $H_1(D)$ is embedded into the scale of H_p spaces, $0 < p \leq \infty$ is receiving very little attention. Let us remark here that the scale of H_p spaces, $0 < p \leq \infty$ seems to be nicer than the much more investigated scale of L_p spaces, $0 < p < \infty$. In particular the passage from $p \geq 1$ to $p < 1$ is much more natural for H_p than for L_p spaces. The generalisation from Banach space case to p-Banach spaces, $p < 1$ is much easier to understand if one thinks in terms of H_p-spaces. Also the dual and predual of $H_1(D)$ receives relatively little attention. This is not intended to mean that the space BMO deserves smaller attention than $H_1(D)$ does.

Now some explanation about the style of exposition. We usually do not give proofs of presented results, however we did our best to give detailed references to the literature where the proof can be found. When the proof is given, it is usually a sketch. As a rule this happens when this particular proof or result is not explicitly stated in the literature, but is an easy consequence of known results or methods.

The paper is divided into seven sections of very unequal length. The first two give the necessary analytic background, the next four are devoted to the subject proper of our exposition while the last one formulates some additional directions for possible future research.

Now we indicate the content of particular sections. Section 1 describes the space $H_1(D)$ as a space of analytic functions on the unit disc while Section 2 presents the real variable approach to the same space. Section 3 gives some results on general subspaces of $H_1(D)$. Short Section 4 is devoted to isometric problems. Section 5 contains the description of bases and unconditional bases in H_1 and closely related results about isomorphisms between $H_1(D)$ and various other spaces. Finally Section 6 discusses complemented subspaces of $H_1(D)$.

SECTION 1;

Complex function approach to $H_1(D)$.

It is well known that the function $f(z)$ analytic in $D = \{z \in \mathbb{C} : |z| < 1\}$ can have an extremally irregular behaviour as $|z|$ approaches 1. So it is a natural idea to consider functions with somehow restricted behaviour close to the boundary. Hardy spaces is one such possibility, which turned out to be extremely successful. For $0 < p \leq \infty$ we define $H_p(D)$ as the space of all analytic functions $f(z)$, $|z| < 1$ such that

$$\sup_{r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < \infty.$$

So we restrict the mean growth of a function. The first important property of a function from $H_p(D)$ is that it leaves a good trace on T. This is summarised in the following

THEOREM 1.1.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H_p(D)$ and let $f_r(e^{i\theta})$ be defined as $f(re^{i\theta})$ for all $r<1$. Then there exists a function $f(e^{i\theta}) \in L_p(T)$ such that

a) $\lim_{r\to 1} f_r = f$ almost everywhere

and in $L_p(T)$

b) $||f||_{L_p(T)} = \sup_{r<1} ||f_r||_{L_p(T)}$

c) if $p \geq 1$ then $f(e^{i\theta})$ has the Fourier series $\sum_{n=0}^{\infty} a_n e^{in\theta}$.

These are by no means trivial facts.

From now on we will very often identify $H_p(D)$ with a subspace of $L_p(T)$. Hidden in those statements is an important F.M.Riesz Theorem. Let us discuss it in more detail. Consider $p=1$. Then $(f_r)_{r<1}$ is a bounded family in $L_1(T)$. If we look at the Fourier series of those functions we easily see that (f_r) converges in the *weak topology to a <u>measure</u> whose Fourier series is $\sum_{n=0}^{\infty} a_n e^{in\theta}$. That this measure is actually an $L_1(T)$ function is a content of the

THEOREM 1.2. (F.M.Riesz)

Let μ be a measure on T such that

$$\int_0^{2\pi} e^{-in\theta} d\mu(\theta) = 0 \quad \text{for} \quad n = 1,2,3,\ldots$$

Then μ is absolutely continuous with respect to the Lebesgue measure.

A functional analytic corollary of this theorem is that $H_1(D)$ is a *weak closed subspace of $M(T)$, so it is a dual space. Actually we can say a little more. Let A_0 denote the closure in $C(T)$ of the set $\{e^{in\theta}\}_{n=1}^{\infty}$ and let H_∞^0 denote the $\sigma(L_\infty, L_1)$ closure of the same set. Those spaces can be identified with the spaces of all analytic functions which are uniformly continuous in D or bounded in D, respectively.

COROLLARY 1.3.

$H_1(D)$ is isometric to the dual of $C(T)/A_o$ and $H_1(D)^*$ is isometric to $L_\infty(T)/H_\infty^o$.

Now we will explain the canonical factorization. We first introduce three classes of functions.

(1) Blaschke products. Let $(a_n)_{n=1}^\infty$ be a sequence of numbers from D such that $\sum_{n=1}^{\infty} (1-|a_n|) < \infty$.

Then

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1-\bar{a}_n z}$$

is an analytic function (called Blaschke product) such that

(a) the zeros of $B(z)$ are exactly $(a_n)_{n=1}^\infty$, counting multiplicity,

(b) $|B(z)| < 1$ for $z \in D$ and $|B(e^{i\theta})| = 1$ a.e.

(2) Singular inner functions are the functions of the form

$$S(z) = \exp\left(-\int_0^{2\Pi} \frac{e^{it}+z}{e^{it}-z} d\mu(z)\right)$$

for some positive singular measure μ on T. Basic properties of singular inner functions are

(a) $S(z) \neq 0$ for $z \in D$

(b) $|S(z)| < 1$ for $z \in D$

(c) $|S(e^{i\theta})| = 1$ a.e.

CAUTION:

$S(z)^{-1}$ is not bounded, it does not even belong to any $H_p(D)$.

(3) Outer functions. For $\psi(t)$ defined on T such that $\psi(t) \geq 0$, $\log \psi \in L_1$, $\psi \in L_p$ we define an outer function of the class H_p to be

$$F(z) = e^{i\gamma}\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log\psi(t)\,dt\right)$$

where γ is a real number. It is easy to see that $|F(e^{i\theta})| = \psi(\theta)$. Obviously $F(z)$ has no zeros in D.

Now we are ready to state

THEOREM 1.4. (Canonical factorization theorem).

Every function $f \in H_p(D)$ admits a unique factorization in the form $f(z) = B(z)\cdot S(z)\cdot F(z)$ where B is a Blaschke product, S is a singular inner function and F is an outer function of class H_p. Conversely every such product belongs to $H_p(D)$.

The following easy corollary from the Canonical Factorization Theorem is very useful.

COROLLARY 1.5.

a) Every $f \in H_p(D)$ can be written as $f = h_1 + h_2$ where h_1, h_2 have no zeros in D and $||h_1||,\ ||h_2|| \le 2||f||$.

b) Let $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, then every $f \in H_p(D)$ can be written as $f = g\cdot h$, $g \in H_q(D)$, $h \in H_r(D)$ and $||f||_{H_p(D)} = ||g||_{H_q(D)}\cdot ||h||_{H_r(D)}$; in particular for $p = 1$ we can have $r = q = 2$.

The proofs are so easy that we can give them here. Let $f = B\cdot S\cdot F$. For a) put $h_1 = (B-1)\cdot S\cdot F$ and $h_2 = S\cdot F$. For b) put $g = B\cdot S\cdot F^{p/q}$ and $h = F^{p/r}$.

The special case of b) that every $H_1(D)$ function is a product of two $H_2(D)$ functions opens up a new description of dual and predual of $H_1(D)$. I do not want to go into details, here. Let me say only that it is possible to represent the predual of $H_1(D)$, the space $C(T)/A_o$, as a space of so called compact Hankel operators on a Hilbert space. The proof of this assertion can be found in [K - P] or [Sar] and a detailed exposition of the theory of those operators is given in [H - P] and [Pow].

NOTE.

All facts presented in this Section are classical and can be found in any of the books [Dur], [Ho], [Koo],[Kat],[Zyg].

SECTION 2;

Real variable approach to $H_1(D)$.

The theory sketched in the previous section depended very much on tools (like Blaschke products) which are peculiar to one-dimensional situation, and are impossible to generalise to several variable situation. Thus attempts to generalise the theory required new tools. It is a remarkable fact that those new tools invented for generalisation had also a tremendous impact on the classical theory. In this section we intend to describe this real variable approach to $H_1(D)$. Let us take $f(z) \in H_1(D)$. We know that $f(e^{i\theta}) \in L_1(T)$, and $||f||_{H_1(D)} = \int_T |f(e^{i\theta})| d\theta$. It is also well known that $\mathrm{Re} f(z)$ determines $\mathrm{Im} f(z)$ up to a constant. Moreover it follows from Theorem 1.1 that both $\mathrm{Re} f(e^{i\theta})$ and $\mathrm{Im} f(e^{i\theta})$ exist on T. Since $||f||_{H_1(D)} < \infty$ if and only if both $\mathrm{Re} f(e^{i\theta})$ and $\mathrm{Im} f(e^{i\theta})$ belong to $L_1(T)$ we may say that we are interested in harmonic functions $h(z)$, $z \in D$ such that both h and its harmonic conjugate have boundary values in $L_1(T)$. It is known (cf.[Kat],[Koo],[Zyg]) that for $f \in L_1(T)$ there exists a harmonic extension via the Poisson formula to a harmonic function $f(z)$ and its harmonic conjugate $\tilde{f}(z)$ has boundary values on T given by the principal value of the following improper integral

$$\tilde{f}(e^{it}) = \int_T f(t-\tau) \cot \frac{\tau}{2} \, d\tau .$$

This justifies the following definition:

$\mathrm{Re}H_1(T)$ is the space of all functions $f \in L_1(T)$ such that $\tilde{f} \in L_1(T)$ with the norm

$$||f||_{\mathrm{Re}H_1(T)} = ||f||_{L_1(T)} + ||\tilde{f}||_{L_1(T)}$$

This space $\mathrm{Re}H_1(T)$ basically consists of real valued functions but there is no difficulty in considering complex functions as well.

Since it is known that for $f \in L_p(t)$, $1 < p < \infty$ $\tilde{f}$ also is in $L_p(T)$ we infer that

$$\bigcup_{p>1} L_p(T) \subset \mathrm{Re}H_1(T) \subset L_1(T).$$

Up till now it may be seen merely as a translation but the point is that in this way we put our space $H_1(D)$ into entirely new perspective. The fundamental development in this context is the Fefferman duality theorem. We start by introducing the space BMO of functions of bounded mean oscillation. Let $f \in L_1(T)$. For every interval $I \subset T$ we put $f_I = \frac{1}{|I|} \int_I f$. We say that $f \in \mathrm{BMO}(T)$ if

$$\sup_{I \subset T} \frac{1}{|I|} \int_I |f - f_I| < \infty.$$

The quantity above is not a norm (it is zero for the constant function) so we define the norm by the formula

$$||f||_{\mathrm{BMO}} = \int_T |f| + \sup_{I \subset T} \frac{1}{|I|} \int_I |f - f_I|.$$

It follows from the John-Nierenberg inequality (cf.[J-N],[Ner]) that the above norm is equivalent (for all p, $1 < p < \infty$) to

$$\int_T |f| + \sup_{I \subset T} \left(\frac{1}{|I|} \int_I |f - f_I|^p\right)^{\frac{1}{p}}.$$

In particular $\mathrm{BMO}(T) \subset \bigcap_{p<\infty} L_p(T)$.

The fundamental Fefferman duality theorem asserts

THEOREM 2.1.

The dual of $\mathrm{Re}H_1(T)$ is $\mathrm{BMO}(T)$; more precisely, for every bounded linear functional $x^* \in [\mathrm{Re}H_1(T)]^*$ there exists a unique function $\varphi \in \mathrm{BMO}(T)$ such that for $f \in L_2(T)$ we have

$$x^*(f) = \int_T f(t)\varphi(t)dt$$

and conversely there exists a constant C such that for every $\varphi \in \mathrm{BMO}(T)$ and $f \in L_2(T)$ we have

$$\left|\int f(t)\varphi(t)dt\right| \leq C ||f||_{\mathrm{Re}H_1(T)} \, ||\varphi||_{\mathrm{BMO}(T)}.$$

REMARK.

The reason for invoking $f \in L_2(T)$ in this statement is that $f(t)\cdot\varphi(t)$ for $f \in ReH_1(T)$ and $\varphi \in BMO(T)$ need not be Lebesgue integrable.

The above Theorem 2.1. implies the following description of $H_1(D)^*$.

THEOREM 2.2.

$H_1(D)^*$ can be identified with the space $BMOA = H_1(D) \cap BMO(T)$.

Now let us explain another equivalent form of the Duality Theorem. We will call the function $a(t)$ defined on T an atom if

either $a(t) \equiv 1$ or

for some interval $I \subset T$ we have

$$\operatorname{supp} a(t) \subset I \quad \text{and} \quad |a(t)| \leq \frac{1}{|I|} \quad \text{and} \quad \int_T a(t)dt = 0.$$

THEOREM 2.3.

A function $f(t)$ belongs to $ReH_1(T)$ if and only if $f = \Sigma\lambda_j a_j$ where all the a_j's are atoms and $\Sigma|\lambda_j| < \infty$. Moreover $\inf \Sigma|\lambda_j|$ over all atomic representations of f is a norm equivalent to $||f||_{ReH_1(T)}$.

PROOF.

An easy calculation shows that $||\tilde{a}||_{L_1(T)} \leq C$ for all atoms, so all atoms have uniformly bounded norm in $ReH_1(T)$. To show the other inclusion it is enough to show that the set of all atoms norms BMO(T). This follows immediately from the observation that mean zero functions in L_∞ of norm ≤ 1 norm mean zero functions in L_1.

Two comments about Theorem 2.3 are in order.

a) This is potentially a very useful theorem. It gives a kind of "extreme point" description of $ReH_1(T)$. Obviously in order to establish the continuity of an operator defined on $ReH_1(T)$ it is enough to check that it is uniformly bounded on all atoms. Since atoms are rather simple functions this task should be much easier than the initial problem.

b) The notion of an atom is very easy to generalise. In particular we can consider atoms on the interval and we can define the space $H_1^a(0,\Pi)$ as the space of all functions f defined on $[0,\Pi]$ such that $f = \Sigma\lambda_j a_j$ where $\Sigma|\lambda_j| < \infty$ and a_j are atoms defined on $[0,\Pi]$. The norm in $H_1^a(0,\Pi)$ is by definition $\inf \Sigma|\lambda_j|$ over all atomic representations of f.

Now we establish an easy proposition which will be used in Section 5. For a function f defined on $[0,\Pi]$ we put

$$\varepsilon f(t) = \begin{cases} f(t) & t\in[0,\Pi] \\ f(-t) & t\in[-\Pi,0]. \end{cases}$$

PROPOSITION 2.4.

For $f \in H_1^a(0,\Pi)$ we define $Tf = \varepsilon\mathrm{Re}f - \varepsilon\mathrm{Im}f + i(\varepsilon\mathrm{Re}f + \varepsilon\mathrm{Im}f)$. The operator T establishes an isomorphism between $H_1^a(0,\Pi)$ and $H_1(D)$.

PROOF.

It is easy to check that T is a complex linear map and that Tf is an analytic function. For a an atom on $[0,\Pi]$ εa is a sum of two atoms on the circle. Moreover "$\sim$" is continuous on $\mathrm{Re}H_1(\mathbb{T})$, so $\varepsilon\mathrm{Re}f - \varepsilon\mathrm{Im}f$ is in $\mathrm{Re}H_1(\mathbb{T})$ thus T is continuous. To check that T^{-1} is continuous it is enough to consider functions of the form $h = a + i\tilde{a}$ where a is a real atom on the circle. We have $h = Tf$ where $\varepsilon\mathrm{Re}f - \varepsilon\mathrm{Im}f = a$. The norm of f is easy to estimate if one remembers that $\varepsilon\mathrm{Re}f$ is even and $\varepsilon\mathrm{Im}f$ is odd.

NOTES. There are many proofs of Theorem 2.1. in the literature. The theorem was announced by C.Fefferman in [Fe] and the proof in the context of $\mathbb{R}^n$ appeared in [F-S]. The proofs for the unit disc can be found in [Koo] and [Sar]. The John-Nierenberg inequality mentioned before Theorem 2.1. was proved in [J-N]. The simpler proofs can be found in [Ner] or [Koo]. The fact that Theorem 2.1. is equivalent to Theorem 2.3 was observed by C.Fefferman (unpublished). Direct (i.e. without use of the duality theorem) proofs of Theorem 2.3. can be found in [Co] and [Wi]. A very general extension of the H_p-theory using atoms is presented in [C-W].

SECTION 3;

Stein's theorem and its consequences.

We start with the formulation of the E.M. Stein multiplier theorem.

THEOREM 3.1.

Let $(\mu(n))_{n=0}^{\infty}$ be a sequence of complex numbers such that

1) $\sup_n |\mu(n)| \leq C$

2) $\sup_n (n+1)\cdot|\mu(n+1)-\mu(n)| \leq C$.

Then the map $\sum_{n=0}^{\infty} a_n z^n \longrightarrow \sum_{n=0}^{\infty} a_n \mu(n) z^n$ is a continuous map from $H_1(D)$ into itself and its norm depends only on C.

Now we define multipliers $\Lambda_n, n=1,2,3,\dots$ as follows

$$\Lambda_n(k) = \begin{cases} 1 & 2^{2n} \leq k \leq 2^{2n+1} \\ 0 & k \leq 2^{2n-1} \text{ or } k \geq 2^{2n+2} \\ \text{linear} & \text{otherwise.} \end{cases}$$

We also put

$$\Lambda_0(k) = \begin{cases} 1 & k=0,1,2 \\ \frac{1}{2} & k=3 \\ 0 & k \geq 4 \end{cases}.$$

It is easy to check that $\sum_{n=0}^{\infty} \Lambda_n(k) = 1$ for all k and that for every $\varepsilon_n = \pm 1$ the sequence $\left(\sum_{n=0}^{\infty} \varepsilon_n \Lambda_n(k)\right)_{k=0}^{\infty}$ satisfies the assumptions of Theorem 3.1. with the same C. So we have

COROLLARY 3.2.

For $f \in H_1(D)$ we have $\sum_{n=0}^{\infty} \Lambda_n(f) = f$ and the series is unconditionally convergent.

By a standard application of the Khintchine inequality we obtain

COROLLARY 3.3.

For $f \in H_1(D)$ we have

$$\|f\|_{H_1(D)} \sim \int_T \left(\sum_{n=0}^{\infty} |\Lambda_n(f)|^2\right)^{\frac{1}{2}}.$$

Let us recall the following

DEFINITION 3.4.

A Banach space X has an unconditional finite dimensional expansion of identity if there exists a sequence of finite dimensional operators $(T_n)_{n=0}^{\infty}$ such that for every $x \in X$, $x = \sum_{n=0}^{\infty} T_n(x)$ and the series is unconditionally convergent.

Obviously an unconditional basis is a special example of a finite dimensional expansion of identity (for the definition see Section 5).

Now we have the following

THEOREM 3.5.

Let X be a subspace of $L_1(T)$ with an unconditional finite dimensional expansion of identity. Then X is isomorphic to a subspace of $H_1(D)$.

PROOF.

We have $T_n : X \longrightarrow X$, finite dimensional and $x = \sum_{n=0}^{\infty} T_n(x)$ and the series is unconditionally convergent. This implies that $|||x||| = \int_T \left(\sum_{n=0}^{\infty} |T_n(x)|^2\right)^{\frac{1}{2}}$ is an equivalent norm on X. By a standard perturbation argument we can assume that $\mathrm{Im}T_n \subset \mathrm{span}\{e^{ijt}\}_{j=p_n}^{k_n}$.

We choose integers μ_n and ν_n, ν_n's being all different in such a way that

$$[\mu_n + p_n, \mu_n + k_n] \subset [2^{2\nu_n}, 2^{2\nu_n+1}].$$

Now it is obvious that the map $x \longrightarrow \sum_{n=0}^{\infty} e^{i\mu_n t} T_n(x)$ is an isomorphic embedding from X into $H_1(D)$.

In particular $H_1(D)$ contains subspaces isomorphic to $L_p(T)$, $1 < p \leq 2$, since L_1 does. Since by a theorem of Rosenthal [Ros] every reflexive subspace of L_1 is isomorphic to a subspace of some L_p, $p > 1$ we get that every reflexive subspace of L_1 is a subspace of $H_1(D)$. This last fact was proved in a much simpler way in [K-P].

As a next application of Stein's theorem we will prove the following result due to S.Kwapien and A.Pelczynski [K-P].

THEOREM 3.6.

Let (f_n) be a sequence in $H_1(D)$ such that $\|\Sigma a_n f_n\|_{H_1(D)} \sim (\Sigma |a_n|^2)^{\frac{1}{2}}$. Then there exists a subsequence (f_{n_k}) such that span (f_{n_k}) is complemented in $H_1(D)$.

PROOF.

By a standard gliding hump argument we can assume that there exist a subsequence (f_{n_k}) and a sequence of multipliers $(\Gamma_k(j))_{j=1}^{\infty}$ such that

a) $\Gamma_k(f_{n_k}) = f_{n_k}$

b) $\Gamma_{k_1}(j) \cdot \Gamma_{k_2}(j) = 0$ for $k_1 \neq k_2$, $j = 0,1,2,\ldots$

c) for all choices of $\varepsilon_k = \pm 1$ the sum $\sum_k \varepsilon_k \Gamma_k(j)$ satisfies the assumptions of Theorem 3.1 with the constant independent of (ε_k).

Let $g_k \in H_1(D)^*$ be such that

1) $\|g_k\| \leq C$

2) $g_j(f_{n_k}) = \delta_{k,j}$

3) $\Gamma_k(g_k) = g_k$.

Now we can define a projection onto span (f_{n_k}) by the formula

$P(f) = \sum_k g_k(\Gamma_k(f)) \cdot f_{n_k}$.

Since

$$\left(\sum_k |g_k(\Gamma_k(f))|^2\right)^{\frac{1}{2}} \leq C\left(\sum_k ||\Gamma_k(f)||^2\right)^{\frac{1}{2}} \leq C\int_T \left(\sum_k |\Gamma_k(f)|^2\right)^{\frac{1}{2}} \leq C||f||$$

we infer that P is a continuous projection onto span (f_{n_k}).

REMARK.

Let us observe that if $f_n = z^{k_n}$ where $k_{n+1}/k_n \geq \lambda > 1$ for all n, then the above argument works without passing to a subsequence, so we obtain the following classical inequality of Paley [Pa].

COROLLARY 3.7.

Let $k_{n+1}/k_n \geq \lambda > 1$ and let $f = \sum_{n=0}^{\infty} a_n z^n$ be in $H_1(D)$. We have

$$\left(\sum_n |a_{k_n}|^2\right)^{\frac{1}{2}} \leq C||f||_{H_1(D)}.$$

REMARK.

Some of the results of this section follow also from the existence of an unconditional basis in $H_1(D)$ (cf. Section 5).

NOTES. Theorem 3.1 and Corollaries 3.2 and 3.3. were proved in [Ste]. A nice proof using atoms can be found in [C-W], Theorem 1.20. Theorem 3.5. is a routine observation which, to the best of our knowledge, does not appear in the literature. Theorem 3.6. was proved in [K-P], Theorem 3.1. The proof given there is different and rather more complicated. We may remark that Theorem 3.5 shows that Theorem 3.6 holds also for $H_1(T^n)$; the H_1 space on the polydisc.

SECTION 4;

Isometric questions.

We already know several isomorphic representations of $H_1(D)$ and we will see many other later. In this situation the importance of isometric theory is rather diminished. On the other hand there are several interesting results and open problems pertaining to the isometric theory of $H_1(D)$ (in this particular norm), so we decided to present them in our exposition. The existing isometric results concentrate on two questions:

a) description of isometries
b) extreme point structure.

The isometries have been described for H_p-spaces on more general domains in $\mathbb{C}^n$ than the unit disc (cf.[Ru 1] and the bibliography given there). The theorem for $H_1(D)$ reads as follows:

THEOREM 4.1.

Every isometry I of $H_1(D)$ into $H_1(D)$ is given by

$$If(z) = g(z)\cdot f(\emptyset(z))$$

where $\emptyset$ is a non-constant inner function in D, $g \in H_1(D)$ and $\int_T b(t)dt = \int_T (b_o\emptyset)(t)|g(t)|dt$ for all bounded, Borel functions $b(t)$ defined on T. Conversely every map described above is an isometry from $H_1(D)$ into itself.

This theorem was proved by F.Forelli [Fo]. For earlier results the reader may consult [Fo],[Ru3],[Ho]. It is known that a subspace of L_p, $p \geq 1$ isometric to L_p is norm one complemented. As far as we know the complementation of subspaces of $H_1(D)$ isometric to $H_1(D)$ was not investigated in general. Some partial results are contained in [Ba] . The other open problem connected with isometries is the following: does $H_1(D)$ contain a subspace isometric to $H_p(D)$, $1 < p \leq 2$? It is known (cf. Theorem 3.5. and following remarks) that $H_p(D)$, $1 < p \leq 2$ is isomorphic to a subspace of $H_1(D)$.

The existing information about extreme point structure of $H_1(D)$ is presented in [Ho], Chapter 9. Let us quote the description of extreme points.

THEOREM 4.2.

Let f be a function in $H_1(D)$. Then f is an extreme point of the unit ball of $H_1(D)$ if and only if f is an outer function of norm 1.

We do not know about any further results. In particular strongly exposed points or denting points of the unit ball in $H_1(D)$ are not described. For definitions of those notions and their importance the reader may consult [D-U]. The problem of description of strongly exposed points is discussed in [GaJ], IV.5. It follows from the general theory (cf.[D-U]) that $H_1(D)$ has many points of both kinds.

The last isometric result we want to mention is the following theorem of Newman's [New].

THEOREM 4.3.

Let $(f_n)_{n=1}^{\infty}$ and f be in $H_1(D)$. Assume that f_n tends to f weakly and $||f_n|| \longrightarrow ||f||$. Then f_n tends to f in norm.

Newman named this property "pseudo-uniform convexity". In the Banach space theory this is expressed as "the Kadec-Klee norm". Kadec and Klee have shown that every Banach space can be renormed to have the property described in Theorem 4.3. For a discussion of this notion in the framework of general Banach spaces see [Di].

SECTION 5;

Bases and various isomorphic representations.

For an arbitrary Banach space X the system of vectors $(x_n)_{n=0}^{\infty} \subset X$ is called a Schauder basis if for every element $x \in X$ there exists a unique sequence of scalars $(a_n)_{n=0}^{\infty}$ such that the series $\sum_{n=0}^{\infty} a_n x_n$ converges to x in the norm of X.

It is well known that in this case each coefficient a_n is actually given by a linear functional $x_n^* \in X^*$, so the series has the form $\sum_{n=0}^{\infty} x_n^*(x)\, x_n$. It is also known that if we are given a biorthogonal system $(x_n, x_n^*)_{n=0}^{\infty} \subset X \times X^*$, i.e. a system such that $x_n^*(x_n) = 1$ and $x_n^*(x_m) = 0$ for $n \neq m$, then such a system is a basis for X if and only if the closed linear span of $\{x_n\}_{n=0}^{\infty}$ equals X and the family of partial sum operators $P_N(x) = \sum_{n=0}^{N} x_n^*(x) x_n$ is uniformly bounded. If the basis (x_n) has the additional property that for every $x \in X$ the series $\sum_{n=0}^{\infty} x_n^*(x)\, x_n$ converges unconditionally, we call such a basis unconditional. The basis is called monotone if $||P_N|| = 1$ for

$N = 0,1,2,\ldots$ and unconditionally monotone if for every $x = \sum_{n=0}^{\infty} x_n^*(x) x_n$ and for every sequence of numbers ε_n with $|\varepsilon_n| = 1$ we have $\|\sum_{n=0}^{\infty} \varepsilon_n x_n^*(x)\, x_n\| \leq \|x\|$.

The problem of the existence of a Schauder basis in $H_1(D)$ was around for quite a long time. It was solved by P.Billard in [Bi] who constructed a basis for $H_1(D)$. His proof was real variable in spirit and used the Haar functions. The general scheme is as follows. Let $(f_n)_{n=0}^{\infty}$ be an orthonormal system on $[0,\Pi]$ with $f_0(t) = \frac{1}{\sqrt{\Pi}}$. Let us define (as in Proposition 2.4.)

$$\varepsilon f_n(t) = \begin{cases} f_n(t) & t \in [0,\Pi] \\ f_n(-t) & t \in [-\Pi,0]. \end{cases}$$

Clearly $(\varepsilon f_n)_{n=0}^{\infty}$ is an orthogonal system on T. In order to obtain a (complex) orthogonal system of analytic functions on T we define

(1) $F_n(z) = \varepsilon f_n(z) + i\widetilde{\varepsilon f}_n(z), \quad |z| < 1.$

The theorem of Billard [Bi] says that if we start with the Haar system on $[0,\Pi]$ as $(f_n)_{n=0}^{\infty}$ then $F_n(z)$ is a Schauder basis in $H_1(D)$. Let us recall that the Haar system on $[0,\Pi]$ is defined as follows.

$$h_0(t) = \frac{1}{\sqrt{\Pi}},$$

if $n = 2^k + j$, $0 \leq j < 2^k$, $k = 0,1,2,\ldots$

then

$$h_n(t) = \begin{cases} -\frac{1}{\sqrt{\Pi}}\, 2^{k/2}, & j\Pi 2^{-k} < t < (2j+1)\Pi 2^{-k-1} \\ \frac{1}{\sqrt{\Pi}}\, 2^{k/2}, & (2j+1)\Pi 2^{-k-1} < t < (j+1)\Pi 2^{-k} \\ 0 & \text{otherwise.} \end{cases}$$

It can be seen from Proposition 2.4. that the theorem of Billard can be formulated as follows.

THEOREM 5.1. (Billard).

The Haar system $(h_n)_{n=0}^{\infty}$ is a basis in $H_1^a(0,\Pi)$.

PROOF.

Obviously the linear span of the Haar system equals $H_1^a(0,\Pi)$. The partial sum operator P_N is given as an averaging operator

$$P_N f = \sum_{j=0}^{N} \frac{1}{|I_j|} \int_{I_j} f \, \chi_{I_j}$$

where I_j's are disjoint intervals covering $[0,\Pi]$ and $\frac{1}{2N} \leq |I_j| \leq \frac{2}{N}$. Let us take an atom $a(t)$, supp $a \subset I$. Since $\int P_N a = 0$, $\|P_N a\|_\infty \leq \|a\|_\infty$ and $\text{diam}(\text{supp } P_N a) \leq |I| + 4/N$ we see that for $|I| \geq \frac{1}{2N}$ the function $P_N a$ is a constant multiple of an atom. In the case $|I| < \frac{1}{2N}$ we have $|I \cap I_j| \neq 0$ for at most two j's. This means that $\text{diam}(\text{supp } P_N a) \leq \frac{4}{N}$. Obviously $\int P_N a = 0$. Moreover $\left|\frac{1}{|I_j|} \int_{I_j} a\right| \leq \frac{1}{|I_j|}$ so $\|P_N a\|_\infty \leq 2N$. This shows that also in this case $P_N a$ is a constant multiple of an atom.

It is relatively easy to check (cf. the last Proposition of [K-P]) that the Haar system is not an unconditional basis in $H_1^a(0,\Pi)$. The question of the existence of an unconditional basis in $H_1(D)$ was raised by several mathematicians (cf.[E],[K-P],[Pe]). Before we present the solution to this problem we need to define a new H_1-space, namely $H_1(d)$, the H_1-space connected with the canonical dyadic martingale. The general theory of this space can be found in [Gar]. From our point of view the most convenient definition is the following. For a function f on $[0,\Pi]$ we define its norm in $H_1(d)$ as

$$\|f\|_{H_1(d)} = \int_0^{\Pi} \left[\sum_{n=0}^{\infty} \left| \int_0^{\Pi} h_n(t) f(t)\,dt \right|^2 |h_n(s)|^2 \right]^{\frac{1}{2}} ds$$

The space of all f such that $\|f\|_{H_1(d)}$ is finite is denoted by $H_1(d)$. It is known that $L_p[0,\Pi] \subset H_1(d) \subset L_1[0,\Pi]$ for all $p > 1$. Moreover it is obvious that the Haar system is an unconditional basis in $H_1(d)$. The question about the existence of an unconditional basis in $H_1(D)$ was answered by the following theorem of Maurey's [Mau].

THEOREM 5.2.

The spaces $H_1(D)$ and $H_1(d)$ are isomorphic.

Maurey's proof of Theorem 5.2. has one drawback, it is not constructive. This was remedied by Carleson [Ca] and the author [Wo5]. In order to describe the result from [Wo5] we need one more definition. We define points t_n as follows

$t_o = 0$, if $n = 2^k + j$, $k = 0,1,2,\dots$, $0 \le j < 2^k$; $t_n = (j+1)2^{-k}\Pi$.

The Franklin system $f_n(t)$ is defined by the following conditions:

$f_o(t) = \frac{1}{\sqrt{\Pi}}$,

$f_n(t)$ is a piecewise linear function on $[0,\Pi]$ with nodes at points $t_o, t_1, \dots, t_n$ which is orthogonal to all f_j's, $j < n$ and $||f_n||_2 = 1$.

We have the following theorem proved in [Wo5].

THEOREM 5.3.

The operator T defined by $T(f_n) = h_n$ extends to an isomorphism from $H_1^a(0,\Pi)$ onto $H_1(d)$. If $F_n(z)$ is defined as in (1) then the map $F_n(z) \longrightarrow h_n$ induces the isomorphism between $H_1(D)$ and $H_1(d)$. In particular the Franklin system is an unconditional basis in $H_1^a(0,\Pi)$ and the corresponding system $F_n(z)$ is an unconditional basis in $H_1(D)$.

It is clear that only the first assertion needs a proof. This can be found in [Wo5] and [Cie] or in a greater generality in [Wo6] or [S-S].

The reader may be interested to note that the existence of an unconditional basis for $H_1(D)$ is a phenomenon from the isomorphic theory. The corresponding isometric fact is false as is given in the following

PROPOSITION 5.4.

$H_1(D)$ is not isometric to a subspace of a Banach space with an unconditionally monotone basis.

The outline of the proof can be found in [Wo5].

The following isometric problem concerning $H_1(D)$ is open.

PROBLEM.

a) Does $H_1(D)$ have a monotone Schauder basis?

b) Describe the norm one, finite dimensional projections in $H_1(D)$.

It is quite likely that the answer to a) is negative. I do not know about any norm one, finite dimensional projection in $H_1(D)$ whose rank has dimension greater than 1. It was shown in [Wo2] that any norm one finite dimensional projection in L_1/H_1 is actually one dimensional.

As the reader may have noticed we have dealt with three different H_1 spaces, and all of them turned out to be isomorphic. It is only a tip of an iceberg. There is a vast proliferation of H_1-spaces important in analysis, cf. [C-W] or [Fo-S]. To decide exactly which are isomorphic to $H_1(D)$ and which are not may be of interest. Maurey in [Mau] considers a wide class of martingale H_1-spaces associated with different sequences of σ-fields and shows them to be isomorphic to $H_1(d)$. He also shows that the Fefferman-Stein spaces $H_1(R^n)$ (for definitions see [F-S]) are isomorphic to $H_1(d)$. If we adhere to the spirit of complex function theory the picture becomes more complicated.

It is possible to consider H_1-spaces of functions analytic on relatively wild sets in $\mathbb{C}$ (cf. [Dur], Chapter 10) but instead we prefer to discuss the situation in several complex variables. There are at least two extremely nice subsets of $\mathbb{C}^n$, the ball B_n and the polydisc D^n. Both admit natural H_1 spaces, which can be defined as a closure of analytic polynomials in a certain norm. To get $H_1(D^n)$ we take the norm

$$||f||_{H_1(D^n)} = \int_{T^n} |f(\xi)|dv(\xi)$$

where v is a normalised invariant measure on a torus T^n. The space $H_1(B_n)$ is obtained if we take as a norm

$$||f||_{H_1(B_n)} = \int_S |f(\zeta)|d\sigma(\zeta)$$

where σ is a normalised, rotation-invariant measure on the unit sphere S.

About those spaces we have the following

THEOREM 5.5.

(a) $H_1(B_n)$ is isomorphic to $H_1(D)$, $n=1,2,3,\ldots$ [Wo7].

(b) If $H_1(D^n)$ is isomorphic to $H_1(D^m)$ then $n=m$ [Bou1],[Bou2].

Both those facts are quite complicated to prove. Let me remark only that in both cases the isomorphism between $H_1(D)$ and $H_1(d)$ is a vital part of the proof.

We feel that a lot remains to be done in connection with this Theorem. In particular we do not know what is the Banach-Mazur distance between $H_1(B_n)$ and $H_1(D)$. Also we do not know what is the situation for other nice domains in $\mathbb{C}^n$. We feel that it should be possible to extend (a) to all strictly pseudoconvex domains. The methods of [Wo8] give many domains which are not strictly pseudoconvex for which (a) holds. The other class of domains which admit nice H_1-spaces are bounded homogeneous domains. Those are fully classified and rather well understood, cf. [Hua],[H-M]. It is a very interesting problem to investigate and classify H_1-spaces on those domains. Some interesting partial results in this direction have been obtained by T. Wolniewicz [Wol].

Let me discuss briefly the method of proof of Theorem 5.5. (b) (cf. [Bou 1]) since it gives a very interesting information about $H_1(D)$. We will formulate it in terms of the Haar system in $H_1(d)$. Let $H_1^N(d)$ denote the span of the first N Haar functions. We say (following J.Bourgain [Bou1]) that a map $\xi : H_1^N(d) \longrightarrow H_1(d)$ is order inversing if $\xi(h_n) = \sum_{k\in A_n} a_k h_k$ and the A_n's are finite, pairwise disjoint sets with $\min A_k > \max A_s$ whenever $k<s$. It is an easy exercise (cf.Theorem 3.5.) that there exist order inversing isomorphic embeddings of $H_1^N(d)$ into $H_1(d)$ with uniform constants. The

main result of [Bou1] asserts however that every projection onto the range of such order inversing isomorphisms has big norm. More precisely: for every $C > 1$ there exists a function $\varphi(N) \xrightarrow[N\to\infty]{} \infty$ such that for every order inversing isomorphism $\xi : H_1^N(d) \longrightarrow H_1(d)$ with $\|f\| \leq \|\xi f\| \leq C\|f\|$, $f \in H_1^N(d)$, the norm of every projection from $H_1(d)$ onto $\xi(H_1^N(d))$ is greater than $\varphi(N)$.

We wish to conclude this section with the description of a recent beautiful and deep result of P.Jones [Jon]. Let us start with the definition of the uniform approximation property.

A Banach space X is said to have the uniform approximation property (cf.[P-R]) if there exist a constant C and a function $\varphi(N)$ such that for every finite system of vectors $x_1,\dots,x_N$ in X there exists a finite dimensional operator $T : X \longrightarrow X$ such that

(1) $Tx_i = x_i$, $i = 1,2,\dots,N$

(2) $\|T\| \leq C$

(3) $\dim T(X) \leq \varphi(N)$.

This is a property stronger than the bounded approximation property since the bound on the dimension of the range of an operator T is imposed.

P.Jones proved in [Jon] that BMO has the uniform approximation property. It was previously unknown (cf.[Pel]) whether BMO has the approximation property. Since uniform approximation property is a self-dual property (i.e. X has it if and only if X^* has it), cf. [Hei], we see that $H_1(D)$ has the uniform approximation property.

SECTION 6;

Complemented subspaces.

In this section we are interested in complemented subspaces of $H_1(D)$. Let us remind that a subspace $X \subset H_1(D)$ is complemented if there exists a projection (an idempotent map) with a range equal to X. Generally, being a complemented subspace of a given space is a much stronger restriction then merely being a subspace. To illustrate this let me remark that for every p, $1 \leq p \leq 2$ l_p is isomorphic to a subspace of $H_1(D)$ (cf. Remarks after Theorem 3.5.) but only l_1 and l_2 are isomorphic to complemented subspaces of $H_1(D)$. This is immediate from Corollary 2.1. of [K-P] which says that the only reflexive complemented subspace of $H_1(D)$ is l_2.

We start however with a more harmonic-analytic problem; what are the projections in $H_1(D)$ which commute with rotations? It is easy to see that such a projection is determined by a subset A of natural numbers, and is given by the formula $\sum_{n=0}^{\infty} a_n z^n \longrightarrow \sum_{n\in A} a_n z^n$. To characterise the idempotent sets, i.e. sets A for which it is a continuous map on $H_1(D)$ is an interesting unsolved problem. The following remarks are obvious:

(1) each finite set is idempotent

(2) if A and B are idempotent then $A \cup B$, $A \cap B$ and $N \setminus B$ are idempotent

(3) if A is idempotent then for every $k \in N$ the translate $(A-k) \cap N$ is idempotent.

(4) a periodic sequence is idempotent.

Let us remark that the above facts (except (3)) are valid also for $L_1(T)$. In this case the idempotent sets have been described by Helson [Hel] as periodic sequences mod a finite set.

For $H_1(D)$ the situation is different. By the Paley theorem (Corollary 3.7.) there is an invariant projection onto a Hilbert space. The theorem of Rudin [Ru4] describes the invariant projections onto Hilbert spaces in $H_1(D)$ as given by a finite sum of lacunary sets. The above mentioned theorem of Kwapien and Pelczynski ([K-P], Corollary 2.1.) say that every complemented reflexive subspace of $H_1(D)$ is isomorphic to a Hilbert space.

The above mentioned facts lead to the following conjecture.

If A is an idempotent set then there exist a periodic set B and lacunary sets (i.e. sets satisfying the assumptions of Corollary 3.7.) $(C_j)_{j=1}^{k}$ and $(D_j)_{j=1}^{s}$ such that

$$A = B \cup \left(\bigcup_{j=1}^{k} C_j \right) \setminus \left(\bigcup_{j=1}^{s} D_j \right).$$

Let us remark that invariant projections for $H_p(D)$, $p<1$ have been described in [K-T] as corresponding to a periodic set mod a finite set. For the invariant, norm one projections on $H_1(D)$ the situation is clear. We have

PROPOSITION 6.1.

There is a constant $c>1$ such that if P is an invariant projection on $H_1(D)$ with $\|P\|<c$ then P is given by a set A of the form $A=\{a+bk,\ k=0,1,2,\ldots\}$ for some $a<b$.

PROOF.

If the set A is not of this form then there exist integers p,q, $p\geq q$ such that $p\in A$ and exactly one of $p+q$ and $p-q$ is in A. So we infer that there exists an invariant projection of norm $<c$ from span $\{z^{p-q},z^p,z^{p+q}\}$ onto span $\{z^{p-q},z^p\}$ or onto span $\{z^p,z^{p+q}\}$. Both those cases reduce to the natural projection from span$\{\bar{z},1,z\}$ onto span $\{\bar{z},1\}$. It is an elementary calculation to show that the norm of this projection is strictly greater than 1.

Now we turn to other class of distinguished subspaces of $H_1(D)$, namely to subspaces invariant for multiplication by z. The celebrated Beurling theorem, specified to $H_1(D)$ (cf.[Dur], Sect.7.3. or [Koo], IV.E.) reads as follows: Let $X\subset H_1(D)$ be such that $z\cdot X\subset X$. Then there exists an inner function I (i.e. a product of a Blaschke product and a singular inner function) such that $X=I\cdot H_1(D)$.

The following theorem holds

THEOREM 6.2.

The multiplication invariant subspace $X=I\cdot H_1(D)$ is complemented in $H_1(D)$ if and only if I is a Blaschke product whose zeros form a Carleson sequence.

Let me recall that a sequence $(z_n)\subset D$ is a Carleson sequence if the measure giving each of the z_n's the mass $(1-|z_n|^2)$ is a Carleson measure.

The proof of this Theorem is almost the same but simpler than the one given for the analogous result in the disc algebra case in [C-P-S].

Now we turn to the (possibly open ended) problem of classifying infinite dimensional complemented subspaces of $H_1(D)$. To do it for a given Banach space is one of the favorite occupations in the geometry of Banach spaces. The case L_p, $p > 1$ has generated a vast literature, the results of [B-R-S] indicate that such a classification in this case has to be very complicated or is impossible at all. In contrast, the case L_1 seems to be much simpler; only two isomorphic types of complemented subspaces of L_1 are known, namely l_1 and L_1. It is a well known open problem if that is all.

Our aim now is to write down all isomorphic types of complemented subspaces of $H_1(D)$ we know of and to indicate that they are indeed different.

We have three building blocks;

(a) by Paley's theorem (Corollary 3.7.) l_2 is isomorphic to a complemented subspace of $H_1(D)$

(b) l_1 is isomorphic to a complemented subspace of $H_1(D)$ (this is very easy)

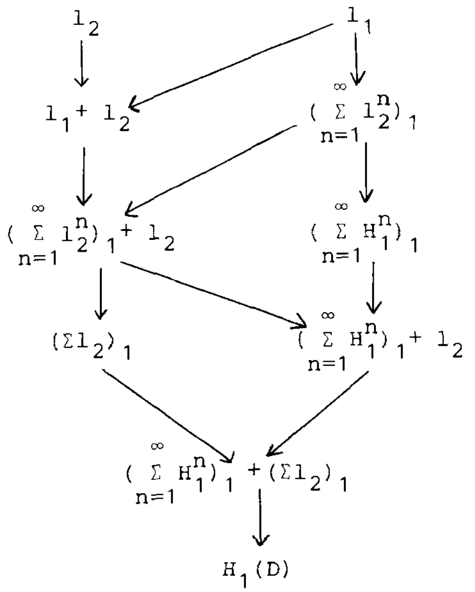
(c) $H_1(D)$ has an unconditional basis; fix a Franklin basis and let H_1^n denote the span of the first n elements of this basis.

We will put those blocks together with the aid of the following

THEOREM 6.3. $H_1(D)$ is isomorphic to its l_1-sum $(\Sigma H_1(D))_1$.

The proof can be found in [Wo3]. Also the isomorphism $H_1(D) \sim H_1(d)$ provides the alternative proof, since for $H_1(d)$ the theorem is easy.

THEOREM 6.4. The following spaces are complemented in $H_1(D)$:



The arrow $X \longrightarrow Y$ means that X embeds into Y as a complemented subspace. All those ten spaces are pairwise non-isomorphic.

PROOF.

Everything except the last statement follows from previous comments. The proof of the last claim uses a lot of Banach space theory but otherwise is quite boring. Nevertheless let us give some hints. We start from the top.

(1) l_2 is the only reflexive space on the list

(2) l_1 is the only one which does not contain l_2^n's uniformly complemented

(3) $l_1 + l_2$ has a unique up to permutation unconditional basis [E-W], which is clearly different from all other spaces

(4) $(\sum_{n=1}^{\infty} l_1^n)_1$ also has a unique up to permutation unconditional basis (cf. [B-C-L-T]) so is not isomorphic to $(\sum_{n=1}^{\infty} H_1^n)_1$

(5) $(\sum_{n=1}^{\infty} H_1^n)_1$ does not contain l_2

(6) $(\sum_{n=1}^{\infty} l_2^n)_1 + l_2$ does not contain $(\Sigma l_2)_1$ and does not contain a complemented copy of $(\sum_{n=1}^{\infty} H_1^n)_1$ (Assume it does, then by [Wo1] $(\sum_{n=1}^{\infty} l_2^n)_1$ would contain $(\sum_{n=1}^{\infty} H_1^n)_1$ complemented, so both would be isomorphic and it is not so.)

(7) since every operator from l_2 into $(\sum_{n=1}^{\infty} H_1^n)_1$ is compact, by [E-W] we obtain that every unconditional basis in $(\sum_{n=1}^{\infty} H_1^n)_1 + l_2$ splits into a part spanning l_2 and a part spanning $(\sum_{n=1}^{\infty} H_1^n)_1$, but the natural basis in $(\sum_{n=1}^{\infty} H_1^n)_1 + (\Sigma l_2)_1$ does not have this property

(8) $(\Sigma l_2)_1$ has a unique up to permutation unconditional basis, see [B-C-L-T]

(9) $(\sum_{n=1}^{\infty} H_1^n)_1 + (\Sigma l_2)_1$ does not contain a subspace isomorphic to l_p, $1 < p < 2$, but $H_1(D)$ has such subspaces.

REMARK.

The isomorphic type of the space $(\sum_{n=1}^{\infty} H_1^n)_1$ does not depend on a choice of basis.

This is a standard decomposition argument.

The very natural question raised by the above discussion is to construct more complemented subspaces of $H_1(D)$. In view of the multitude of apparently different isomorphic representations for $H_1(D)$ (see Section 5) we believe that such a construction is possible.

It is still an open problem raised in [Cas] if $H_1(D)$ is primary. Let us recall that a Banach space X is primary if for every decomposition $X = X_1 + X_2$ we have at least one of X_1 or X_2 isomorphic to X. It is our belief that the answer to this problem is positive. The following result seems to justify this belief.

THEOREM 6.5.

Let X be a subspace of $H_1(D)$. Assume that X is isomorphic to $H_1(D)$. Than X contains a smaller subspace Y complemented in $H_1(D)$ and isomorphic to $H_1(D)$.

The proof of this theorem is a verbatim repetition of the proof of the analogous statement for L_p, $1 < p < 2$ given in [J-M-S-T], pp.265-70.

Now we will discuss briefly the space of polynomials. To be precise, let P_n denote the $\text{span}\{1, z, \ldots, z^n\}$ in $H_1(D)$. It is well known that in its natural position as a subspace of $H_1(D)$ the space P_n is badly complemented, more precisely the norm of the best projection is of order $\log(n+1)$. This result however depends on the particular position of P_n in $H_1(D)$. It is a natural question (cf.[Wo4]) if the P_n's are isomorphic to uniformly complemented subspaces of $H_1(D)$. A positive answer to this was given by J.Bourgain and A.Pelczynski. Their argument runs as follows: As is well known $H_1(D)$ is isomorphic to $H_1(D) + H_1(D)$. Now we define

$$i_n : P_n \longrightarrow H_1(D) + H_1(D) \quad \text{by}$$

$$i_n(z^k) = (z^k, z^{n-k})$$

and $q_n : H_1(D) + H_1(D) \longrightarrow P_n$ by

$$q_n\left(\sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} b_k z^k\right) = \sum_{k=0}^{n} \frac{n-k}{n} a_k z^k + \sum_{k=0}^{n} \frac{n-k}{n} b_k z^{n-k}.$$

Since $\|i_n\| \leq 2$ and $\|q_n\| \leq 1$ and $q_n \circ i_n = id$ the claim follows.

Actually S.V.Bočkariov [Bo] has shown more. He has proved that P_n is uniformly isomorphic to H_1^{n+1}. This is incomparably more complicated.

SECTION 7;

Concluding Remarks.

In the previous sections the patient reader has found many open problems connected to the results we have been discussing. In this section we want to present some directions for possible further research.

1) Multipliers. It is a well known open problem to give good criteria for a sequence $(\mu(n))_{n=0}^{\infty}$ to be a multiplier from $H_1(D)$ into $H_1(D)$. The Theorem 3.1. is a fine example. Our knowledge however is still quite small as is witnessed by the problem of description of 0,1 valued multipliers (i.e. invariant projections). There is a vast literature on multipliers both from $H_1(D)$ into itself and from $H_1(D)$ into other natural spaces. We do not want to discuss it here. Let us mention only that multipliers from $H_1(D)$ into $H_2(D)$ and from $H_1(D)$ into l_1 are fully described (cf.[Dur], Th.6.4 and [Sl-St], resp.). However, as far as we know, all the existing theorems deal with boundedness or compactness of multipliers. It is our belief that it is worthwhile to investigate what multipliers belong to other natural and important operator ideals. The theory of operator ideals (cf.[Pie]) is a very important tool in various parts of analysis, its applications to the disc algebra are discussed in [Pel]. In this situation we believe that some theorems of this type may turn out to be very useful.

2) Finite dimensional structure. The problem basically is to develop the local theory of $H_1(D)$. In particular to investigate various parameters associated with its main finite dimensional building blocks, i.e. the spaces H_1^n. The presentation of some parts of local theory of Banach space (in the context of classical spaces) can be found in [Pel1]. The only paper which contains some results of this nature connected with $H_1(D)$ is [G-R].

3) Convergent Taylor series. It is well known that the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ of an $H_1(D)$ function f need not converge to f in norm. We feel that the space of convergent Taylor series, i.e. the space consisting of all $f = \sum_{n=0}^{\infty} a_n z^n \in H_1(D)$ such that

$$||f||_u = \sup_N \Big|\Big| \sum_{n=0}^{N} a_n z^n \Big|\Big|_{H_1(D)} < \infty,$$

may be an interesting object to study. I do not know of a single work dealing with this space. The analogous space for $p=\infty$, i.e. the space of uniformly convergent Taylor series received much attention in recent years. The basic reference for this is [Vin]. We feel however that there is a very small connection between $p=1$ and $p=\infty$ in this case.

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Functional Analysis: Surveys and Recent Results III
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GATEAUX DIFFERENTIATION OF FUNCTIONS ON A CERTAIN CLASS OF BANACH SPACES

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This is an expanded version of a lecture given at the Third Paderborn Conference in Functional Analysis, May 1983.

One of the first unpleasant examples that one sees in mathematics is a function of two variables $f(x,y)$ that is nicely differentiable in each coordinate but is not "differentiable". In higher dimensions or even in infinite dimensions things become even worse. Let us relate this to an old and still unsolved problem of Banach: Given an infinite dimensional Banach space X, is X isomorphic (linearly homeomorphic) to a closed hyperplane? In other words, is X isomorphic to $X \times R$, where R is the real numbers? The difficulty of this problem is that of finding a suitable invariant for X. One useful invariant is the following: Suppose X has the property that each continuous convex function on X is (Gateaux) differentiable on a dense set. The special case of Banach's problem is also unsolved: If X has this differentiation property, does $X \times R$?

We begin with the definitions of Gateaux and Fréchet differentiability. Let X and Y be Banach spaces and U an open subset of X. A continuous function $f : U \to Y$ is Gateaux differentiable at a point x_0 in U if there exists an operator (a continuous linear function) $T : X \to Y$ such that for all x in X, $\|x\| = 1$,

$$\lim_{t\to 0} \frac{f(x_0+tx) - f(x_0)}{t} = T(x).$$

Such a function is Fréchet differentiable at an x_0 in U if

$$\lim_{x\to x_0} \frac{\|f(x)-f(x_0) - T(x-x_0)\|}{\|x - x_0\|} = 0.$$

We shall define a Banach space X to be a (weak) Asplund space if every real-valued, continuous and convex function defined on an open subset of X is (Gateaux)

Fréchet differentiable on a dense G_δ subset of that open set. It is a well-known result that finite dimensional spaces are Asplund. Indeed, since there are analytical characterizations of Asplund spaces it is easy to see that the class of Asplund spaces is closed under the operations of finite products, closed linear subspaces and quotient spaces (continuous linear images). Very much less is known about the class of weak Asplund spaces, particularly the permanence properties of this class. It is a result of Mazur that any separable Banach space is weak Asplund, and Asplund proved that any closed linear subspace of a weakly compactly generated Banach space is weak Asplund. Our objective is to construct a class of Banach spaces that (1) contains the Asplund spaces, (2) contains the weakly compactly generated spaces, (3) has reasonable permanence properties and (4) the class is contained in the weak Asplund spaces. Our basic techniques are topological in nature. Our main motivation is the following old and non-trivial theorem. The Fréchet differentiability part is due to Smulian (before 1944) and the Gateaux part is due to many authors, culminating with Mazur in the 1930's.

DIFFERENTIATION THEOREM

Let φ be a continuous and convex function defined on the Banach space X. Then the following are equivalent (we may assume that $\varphi(0) = -1$ and $\varphi(x_0) \neq 0$):

1) The function φ is Gateaux (Fréchet) differentiable at x_0;

2) If $G = \{(x,t) \in X \times \mathbb{R} : t \geq \varphi(x)\}$ is the supergraph of φ, then $(x_0,\varphi(x_0))$ exposes (strongly) G^0, the polar of G;

3) The Minkowski functional of G is Gateaux (Fréchet) differentiable at $(x_0,\varphi(x_0))$;

4) For each $x \in X$, choose $\psi(x) = x^*$ such that $\langle\psi(x),y-x\rangle \leq \varphi(y) - \varphi(x)$ for all $y \in X$ (which exists by the Hahn-Banach theorem); then the function ψ is norm to weak* (norm to norm) continuous at x_0.

PROPERTIES OF PERFECT MAPS

All of our topological spaces will be Hausdorff and completely regular. A topological space is Baire if no non-empty open subset is of the first Baire category.

We recall the definition and elementary properties of perfect maps. A continuous function $f : A \to B$ is perfect if it is onto, closed and has compact fibers, that is $f^{-1}(b)$ is compact for each $b \in B$. A perfect map $f : A \to B$ is minimal if the image of any proper closed subset of A is a proper subset of B.

All of the following elementary facts about perfect maps can be found in [B] (see also [E]).

Let $f : A \to B$ be perfect, then

(Pi) there exists a closed subset A_{min} of A such that $f|A_{min} : A_{min} \to B$ is perfect and minimal;

(Pii) for any subset B_0 of B, $f : f^{-1}(B_0) \to B_0$ is perfect;

(Piii) $f^{-1}(C)$ is compact for each compact subset C of B;

(Piv) if $g : C \to D$ is a homeomorphism onto, then $f \times g : A \times C \to B \times D$ is perfect;

(Pv) the composition of a finite number of perfect maps is perfect;

(Pvi) the product of a finite number of perfect maps is perfect;

(Pvii) if C is compact then $C \times A \to A$ defined by projection is perfect;

(Pviii) if $f : A \to B$ is an onto function then the following are equivalent:

- (a) f is perfect,
- (b) $G = \{(a,b) : f(a) = b\}$ is a closed subset of $\beta(A) \times B$,
- (c) for any homeomorphic embedding $i : A \to K$, K compact, the set $\{(ia,f(a)) : a \in A\}$ is closed in $K \times B$;

(Pix) if $f : A \to B$ and $g : B \to C$ are continuous and onto and $g \circ f$ is perfect then g and f are perfect;

(Px) if $f : A \to B$ is perfect and minimal and B is Baire, then A is Baire;

(Pxi) suppose $f : A \to B$ is perfect, then f is minimal if and only if for every pair a_0, b_0 so that $f(a_0) = b_0$ and every pair of open sets U, V so that $a_0 \in V$, $b_0 \in U$, it follows that $\{b \in U : f^{-1}(b) \subseteq V\}$ is open and non-empty.

(Pxii) suppose $f : A \to B$ is minimal perfect, then for $b_0 \in B$, the following are equivalent:

- (a) there exists a function $g : B \to A$ such that $fg(b) = b$ and g is continuous at b_0;
- (b) for any mapping $g : B \to A$ such that $fg(b) = b$, g must be continuous at b_0;
- (c) $f^{-1}(b_0)$ has only one point;

($\underline{P}$xiii) if $f : A \to B$ is minimal perfect then for any open subset U of B, $f|f^{-1}(U) : f^{-1}(U) \to U$ is minimal perfect.

Although we shall have little to say about set-valued mappings, we shall give a few facts about them for readers who prefer this way of looking at things. We cannot emphasize strongly enough that it is a routine, indeed boring, exercise to reformulate our results in terms of set-valued mappings. Let S and T be topological spaces and $P(S)$ be the power-set of S. A function $F : T \to P(S)$ is upper semi-continuous (usc) if for any open subset U of S, $\{t : F(t) \subseteq U\}$ is an open subset of T. Suppose $F : T \to P(S)$ is usc and each $F(t)$ is a compact and non-empty subset of S. For such a mapping F, define

$$G = \{(s,t) : s \in F(t),\ t \in T\}.$$

Then

$$\text{proj}_T : G \to T$$

is perfect. Conversely, if $f : S \to T$ is a perfect map then

$$F : T \to P(S)$$

defined by $F(t) = f^{-1}(t)$ is usc with compact non-empty values. This is a well known duality. With this duality in mind, we have two choices: we can give a very simple definition of a class of topological spaces which is awkward to apply, or, an awkward definition that is easy to apply. We shall begin with the simple definition:

DEFINITION: $\underline{C}'$ will denote the class of topological spaces with the following property: $K \in \underline{C}'$ if and only if for every Baire space T and for every compact (non-empty) valued usc function

$$F : T \to P(K)$$

there exists a dense G_δ subset G of T and a function $f : T \to K$ such that

$$f(t) \in F(t) \quad \text{for all} \quad t \in T$$

and f is continuous at each point of G.

Unfortunately, this definition is not very useful for the applications we have in mind, so we shall define another class $\underline{C}$ and show that $\underline{C} = \underline{C}'$.

In fact, we only give the definition of $\underline{C}'$ above to appease the topologists who objected to the definition of $\underline{C}$ on aesthetic grounds.

DEFINITION (see [S4]): A topological space K is in $\underline{C}$ if and only if for all topological spaces C,S,T,B so that B is Baire, $C \subseteq K \times S$, all $f : C \to T$ perfect and all $g : B \to T$ continuous there exist a dense G_δ subset G of B and a function $h : B \to K$ such that for each $b \in B$ there exists an $s \in S$ such that $(h(b),s) \in f^{-1}(g(b))$ and h is continuous at each point of G.

The first thing to check is that $\underline{C} \subseteq \underline{C}'$ and this is easy: suppose $F : B \to P(K)$ is usc and compact valued. Define

$$C = \{(k,b) : k \in F(b)\}.$$

As before

$$\mathrm{proj}_B : C \to B$$

is perfect. Of course, g is the identity on B, and we have the desired function $h : B \to K$ and dense G_δ subset G of B.

The converse requires a bit more work: Suppose K is in $\underline{C}'$ and C,S,T,B,f and g are as in the definition of $\underline{C}$. We need to check that the set-valued function $F : B \to P(K)$ defined by $F(b) = \{k : \exists s \in S$ so that $(k,s) \in f^{-1}(g(b))\}$ is usc. This is a routine computation if one uses ($\underline{P}$viii).

It should not be necessary to justify the presence of the space S in the definition of $\underline{C}$, but it is because of some ill-considered commentaries we have received. First, and least important, it makes the verification of ($\underline{C}$iii) below utterly trivial. Most importantly, S does not have to be there, but it may! The space S corresponds to the "dummy" variable. See our penultimate theorem. The definition $\underline{C}$ is glib and hides the important properties of the class of topological spaces under consideration. We now list the permanence properties of the class $\underline{C}$:

($\underline{C}$i) If K_1 is in $\underline{C}$ and $f : K_1 \to K_2$ is perfect then K_2 is in $\underline{C}$;

($\underline{C}$ii) if K_1 is in $\underline{C}$ and $f : K_2 \to K_1$ is continuous and 1:1, then K_2 is in $\underline{C}$ (in particular, if K in the topology τ is in $\underline{C}$ and τ_1 is a finer topology then (K,τ_1) is in $\underline{C}$;

($\underline{C}$iii) $\underline{C}$ is closed under countable products;

($\underline{C}$iv) if $K = \bigcup_{n=1}^{\infty} K_n$, and each K_n is a closed subset of K, and K_n is in $\underline{C}$, then K is in $\underline{C}$;

($\underline{C}$v) if I is a set and $\{K_i\}_{i\in I}$ is a subclass of $\underline{C}$ and each K_i is locally compact then the one-point compactification of the disjoint union is in $\underline{C}$.

The verification of (i) is elementary. To see (ii) let $G \subseteq K_2 \times B$, where B is Baire, $\text{proj}_B(G) = B$, and G is closed in $\beta(K_2) \times B$. Assume proj_B is minimal on G. Then, define $\tilde{f} \times \text{id}_B : \beta(K_2) \times B \to \beta(K_1) \times B$ where $\tilde{f} : \beta(K_2) \to \beta(K_1)$ is the obvious extension. Thus, $\tilde{f} \times \text{id}_B$ is a homeomorphism on G. Let $G_1 = (\tilde{f} \times \text{id}_B)(G)$ and we have that $\text{proj}_B : G_1 \to B$ is minimal perfect (try ($\underline{P}$ix)).

So there exists a dense G_δ set $D \subseteq B$ such that for $b \in D$ there exists a unique $k_2 \in K_2$ such that $(k_2,b) \in G_1$. Since $\tilde{f} \times \text{id}_B$ was 1:1 on G there is a unique $k_1 \in K_1$ such that $(k_1,b) \in G$. Apply ($\underline{P}$xii) to obtain the desired results.

The equivalence of $\underline{C}$ and $\underline{C}'$ yields ($\underline{C}$iii).

To see property ($\underline{C}$iv), let $K = \bigcup_{n=1}^{\infty} K_n$ where each K_n is a closed subset of K, $C \subseteq K \times B$ and $f : C \to B$ perfect, where f is projection. Assume also that $f : C \to B$ is minimal. Since $D_n = C \cap (K_n \times S)$ is closed in C, $f(D_n)$ is a closed subset of B. Since B is a Baire space we have that

$$\bigcup_{n=1}^{\infty} \text{int}\, f(D_n)$$

is a dense subset of B. Since $f : C \to B$ is minimal, $f|f^{-1}(V) : f^{-1}(V) \to V$ is also minimal for an open subset V of T. Hence, by minimality $f^{-1}(\text{int}\, f(D_n))$ is a subset of D_n, and by the remarks above, there exists $g_n : \text{int}\, f(D_n) \to D_n$ continuous, uniquely defined on a dense G_δ subset G_n of $\text{int}\, f(D_n)$ and $f(g_n(b),b) = b$. We may assume $G_n = \bigcap_{m=1}^{\infty} W_{n,m}$, $W_{n,m}$ is an open dense subset of $\text{int}\, f(D_n)$. We know that $g_n = g_m$ on $G_n \cap G_m$ and the g_n's patch together to form a function continuous except perhaps on $\bigcup_n \bigcup_m f(D_n) \setminus W_{n,m}$ which is first category.

To check (v), let $\dot{\bigcup}_i K_i \cup \{\infty\} = K$ be the one-point compactification of the disjoint union $\dot{\bigcup}_i K_i$. Suppose $C \subseteq K$ and $f : C \to B$ is minimal perfect.

On the closed set $B_\infty = \{b : f^{-1}(b) \ni \infty\}$ define

$$g : B_\infty \to K \text{ by } g(b) = \infty.$$

It remains to define g on the complement of B_∞. Let

$$B_i = \{b : f^{-1}(b) \subseteq K_i\}.$$

Each B_i is open and because f is minimal

$$\bigcup_i B_i \cup B_\infty^o$$

is dense in B. For each B_i that is non-empty we may find a function $g_i : B_i \to K_i$ with the desired properties. A function $g : B \to K$ defined arbitrarily (subject to $f(g(b)) = b$) on the compliment of $\bigcup_i B_i \cup B_\infty^o$ and equal to g_i on B_i will have the desired properties on all of B.

We have a reformulation of a result of Mazur:

THEOREM: Let K be a metric space. Then K is in $\underline{C}$.

PROOF: Suppose $C \subseteq K$ and $f : C \to B$ is minimal perfect and B is a Baire space. From the minimality we know that

$$\mathcal{U}_n = \{b \in B : \text{diameter } f^{-1}(b) \leq \frac{1}{n}\}$$

contains an open dense subset of B. That is, each $b \in \bigcap_{n=1}^{\infty} \mathcal{U}_n$ has a one point fiber. Apply ($\underline{P}$xii).

Also, it follows from known results that Eberlein compacts are in $\underline{C}$ (see [S2-4]) and sets with the Radon-Nikodym property in appropriate weak topologies are in $\underline{C}$ (this is one of the main results of [S2] reformulated in the language of perfect maps, see [S4]).

THEOREM: Let K be a subset of the Banach space X and Y a linear subspace of X^* such that every Y bounded subset of K is Y dentable. Then, (K,τ) is in $\underline{C}$ for any topology τ finer than the weak topology on K determined by Y.

We shall define $\underline{S}$ to be the class of all Banach spaces such that X is in $\underline{S}$ if and only if $(X^*,\sigma(X^*,X))$ is in $\underline{C}$. If X is a separable Banach space, $(B(X^*),\omega^*)$ is a compact metric space. Applying ($\underline{C}$iv) we know that X is in $\underline{S}$.

We turn now to the permanence properties of $\underline{S}$:

($\underline{S}$i) if X is in $\underline{S}$ and $T : X \to Y$ is an operator with dense range then Y is in $\underline{S}$;

($\underline{S}$ii) if X is in $\underline{S}$ and $T : Y \to X$ is an operator so that T^{**} is 1:1, then Y is in $\underline{S}$ (in particular, if Y is a closed linear subspace of X, then Y is in $\underline{S}$);

($\underline{S}$iii) $\underline{S}$ is closed under "countable products", for example if X_n is in $\underline{S}$, then $Y = (\sum_{n=1}^{\infty} X_n)_{l_1}$ is in $\underline{S}$, and, using ($\underline{S}$i), a dense image of Y is in $\underline{S}$;

($\underline{S}$iv) if I is a set and X_i is in $\underline{S}$ for each $i \in I$ then $(\sum_{i\in I} X_i)_{c_o}$ is in $\underline{S}$ and for $1 < p < \infty$, $(\sum_{i\in I} X_i)_{l_p}$ is in $\underline{S}$.

A few words about the permanence properties of $\underline{S}$: first, (i) is clear because T^* maps $(B(Y^*),w^*)$ homeomorphically into the ball $B(X^*,\|T\|)$ with the weak* topology. If $(B(Y^*),w^*)$ is in $\underline{S}$, so is (Y^*,w^*) (($\underline{C}$iv)). Property ($\underline{S}$iii) follows from ($\underline{S}$i) and ($\underline{C}$iii). That $\underline{S}$ is closed under arbitrary c_o sums follows from ($\underline{C}$v). Once ($\underline{C}$ii) has been established, it is easy to see that $(\sum_i X_i)_{l_p}$, $1 < p < \infty$, is in $\underline{S}$ because the identity function

$$(\sum_i X_i)_{l_p} \to (\sum_i X_i)_{c_o}$$

satisfies ($\underline{C}$ii). What remains is ($\underline{C}$ii). Suppose $T : Y \to X$ has the property that T^{**} is 1:1 and X is in $\underline{S}$. Suppose C is a subset of Y^*, B is Baire space and

$$\varphi : C \to B \text{ is minimal perfect.}$$

Define $f_n : X^* \times \bar{B}_{Y^*}(0,\frac{1}{n}) \to Y^*$ by $f_n(x^*,y^*) = T^*x^* + y^*$. Since T^{**} is 1:1 we know that T^* has norm dense range and f_n is onto. Let

$$C_m = \{b : \varphi^{-1}(b) \cap f_n[\bar{B}_{X^*}(0,m) \times \bar{B}_{Y^*}(0,\tfrac{1}{n})] \neq \emptyset .$$

Each C_m is closed and $\bigcup_{m=1}^{\infty} C_m = B$.

Define $U_1 = C_1^o$, $U_{m+1} = C_{m+1}^o \setminus \bigcup_{i\le m} C_i$.

Thus, $\bigcup_{m=1}^{\infty} U_m$ is an open dense subset of B. Since $\varphi|\varphi^{-1}(U_m) : \varphi^{-1}(U_m) \to U_m$ is minimal and for $b \in U_m$

$$\varphi^{-1}(b) \cap f_n[\bar{B}_{X^*}(0,m) \times \bar{B}_{Y^*}(0,\tfrac{1}{n})] \neq \emptyset$$

we have that

$$\varphi^{-1}(b) \subseteq f_n[\bar{B}_{X^*}(0,m) \times \bar{B}_{Y^*}(0,\tfrac{1}{n})].$$

Define $A_m = \{(x^*,y^*) : \|x^*\| \le m, \|y\| \le 1/n$ so that $\varphi(T^*x^* + y^*) \in U_m\}$ in the product of the w^*-topologies.

The map

$$\psi_m : A_m \to U_m$$

defined by $\psi_m(x^*,y^*) = \varphi(T^*x^* + y^*)$ is perfect. Use the properties of $\underline{C}$ (here, y^* is the dummy variable) to obtain a function $g_m : U_m \to \bar{B}_{X^*}(0,m)$ so that g_m is continuous at each point of a dense G_δ subset of U_m and for each b in U_m there exists a y^* in Y^*, $\|y^*\| \leq 1/n$, $(g_m(b),y^*)$ in A_m and

$$\psi_m(g_m(b),y^*) = \varphi(T^*g_m(b) + y^*) = b.$$

Define $h_n : B \to X^*$ so that $h_n | U_m = g_m$. For each n, h_n is continuous on a dense G_δ subset of B. Let b_0 be a point of continuity of all h_n. Let y_0^* be a w^* cluster point of $T^*h_n(b_0)$, which must be in $\varphi^{-1}(b_0)$. Let W be a convex symmetric weak* neighbourhood of the origin in Y^*.

Choose p large enough so that $\frac{1}{3}W \supseteq \bar{B}_{Y^*}(0,\frac{1}{p})$. There exists a $q \geq p$ such that $T^*h_q(b_0) \in y_0^* + \frac{1}{3}W$. Since T^*h_q is continuous at b_0 there exists U open, $b_0 \in U$, such that

$$(T^*h_q)(U) \subseteq y_0^* + \tfrac{1}{3}W.$$

Therefore, for $b \in U$,

$$\varphi^{-1}(b) \cap [y_0^* + \tfrac{2}{3}W] \supseteq \varphi^{-1}(b) \cap [\bar{B}_{Y^*}(0,\tfrac{1}{p}) + y_0^* + \tfrac{1}{3}W] \neq \emptyset.$$

The minimality says that $\varphi^{-1}(b)$ is the one point set $\{y_0^*\}$ and we have verified that Y^* in the weak* topology is in $\underline{C}$.

The following is a reformulation of some of the results of [S1-4]:

THEOREM: (a) if X is in $\underline{S}$ then X is a weak Asplund space;
(b) if X is an Asplund space then X is in $\underline{S}$.

From (b), ($\underline{S}$i) and the factorization theorem [DFJP] it follows that all weakly compactly generated spaces are in $\underline{S}$ (indeed the much bigger class of all GSG spaces are in $\underline{S}$, see [S2]). As a very special case, from ($\underline{S}$ii) it follows that if X is wcg and $T : Y \to X$ has property that T^{**} is 1:1, then Y is in $\underline{S}$; in particular, Y is weak Asplund if T is isomorphism (a result of Asplund) (compare also [CK]).

An example of the sort of result that can be obtained by using our techniques combinated with the Differentiation theorem is:

THEOREM: Let X be a Banach space in $\underline{S}$.

(i) Let U be an open subset of the Banach space Z and let $\beta : U \to X$ be continuous and Gateaux differentiable on a dense G_δ subset of U; let $\varphi : X \to \mathbb{R}$ be continuous and convex. Then $\varphi \circ \beta$ is Gateaux differentiable on a dense G_δ subset of U.

(ii) Let S be a topological space such that $X \times S$ is a Baire space; $\varphi : X \times S \to \mathbb{R}$ locally bounded and for each $s \in S$, $\varphi(.,s)$ is continuous and convex. Then there exists a dense G_δ subset G of $X \times S$ such that at each point of G the X-partial Gateaux derivative of φ exists.

For a proof, see [S4].

We shall bore the reader by one foray into "optimization theory". Suppose K is in $\underline{C}$; for f in $C(\beta K)$, define

$$K_f = \{k \in \beta K, \text{ where } f(k) = \rho(f)\}, \quad \rho(f) = \sup\{f : K\} \text{ and}$$
$$A = \{f \in C(\beta K) : K_f \subseteq K\}.$$

Suppose U is an open subset of a Banach space Y and g is a continuous function from U to Y with the properties that there exists a dense G_δ subset G of U so that g is Gateaux differentiable at each point of G and $g(u)$ is in A for u in G. Then the composition $\rho \circ g$ is Gateaux differentiable on a dense G_δ subset of U (this is not a "chain rule"). The proof follows immediately from the definition of $\underline{C}$ and the theorem above. In particular, if K is also compact, $U = C(K)$ and g is the identity function then, by applying the differentiability theorem, we have that

$$\{f \in C(K) : \rho \text{ is Gateaux differentiable at } f\} =$$
$$\{f \in C(K) : f \text{ attains its } \sup \text{ at a single point of } K\}$$

and this set is a dense G_δ subset of $C(K)$. This proves also that if K is compact and in $\underline{C}$ then

$$\{k \in K : k \text{ has a countable basis of neighbourhoods}\}$$

is a dense subset of K and that K is sequentially compact, that is, every sequence in K has a converging subsequence.

Let S be an uncountable set. One easily computes that the norm on $l_1(S)$ is not Gateaux differentiable at any point. Hence, the unit ball of $(l_1(S))^*$ in the weak* topology (which is homeomorphic to the S-fold product of [0,1]) is not in C̲. The topological properties above can also be used to show this and for other spaces as well.

Rather surprising is the following:

THEOREM: If X is in S̲ then $(X^*, \sigma(X^*, X^{**}))$ is in C̲. In particular, l_∞ in its weak topology is in C̲.

An interesting reformulation of an old problem (see [N]) is the following:

PROBLEM: Is any Banach space in its weak topology in C̲ ?

If, in the definition of C̲ one restricts the space B to be complete metric (which is enough for differentiability results) or Čech-complete (or, even strongly countably complete) one obtains new classes such that all Banach spaces in their weak topologies are in these classes. One need only translate the proofs of [N] into our terminology. It involves only the straightforward copying of the interesting results of [N]. Indeed, this has been done several times over in the literature. Our results have obvious reformulations in the language of multivalued maps, monotone operators, convex analysis, optimization, etc.; this would be suitable work for a master's thesis. We would be surprised, however, if these reformulations gave any new insight into the obvious non-trivial problems that have arisen.

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BANACH SPACES OF OPERATORS AND LOCAL UNCONDITIONAL STRUCTURE

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We study here the local unconditional structure of certain operator ideals. Invariants like the Gordon-Lewis constant are investigated.

Grothendieck[6] defined a one absolutely summing operator between two Banach spaces, to be an operator which maps every unconditionally convergent series to an absolutely convergent series. It was asked in [6] whether every one absolutely summing operator can be factored through an $L^1(\mu)$-space. This problem was solved in the negative by Gordon and Lewis [4].

By this they also answered other problems. They observed that every one absolutely summing operator from a Banach space E into another space factors through an $L^1(\mu)$-space provided that E has local unconditional structure (lust). It was unknown whether there are Banach spaces without local unconditional structure. Thus they showed that there are in fact spaces without lust.

The spaces they considered are spaces of operators with an ideal norm [12]. In particular, they found that the space of all bounded operators $L(\ell^2)$ with operator norm does not have local unconditional structure and that there are one absolutely summing operators from $L(\ell^2)$ to ℓ^2 that do not factor through an $L^1(\mu)$-space.

In [15] the proof was simplified and the result was strengthened so that some open questions could be answered.

V. Samarsky [14] proved at the same time independently a weaker result that, nevertheless, could be successfully applied in all interesting cases.

Then Szarek [17] simplified the proof so that it is by now very short. It is completely contained in this paper.

Besides the space $L(\ell^2)$ we consider the Schatten class C_p, $1\leq p<\infty$, all operators from ℓ^2 into itself with p-summable singular numbers [12]. It turns out [8] that C_p has lust if and only if p=2, i.e. C_p

is isomorphic to a Hilbert space.
For the standard notions in Banach space theory we refer to [9].

0. Preliminaries

The Banach-Mazur distance of two isomorphic Banach spaces E and F is given by

$$d(E,F)=\{ \|I\|\|I^{-1}\| \mid I \text{ is isomorphism between } E \text{ and } F\}$$

The space of p-absolutely summing operators A is denoted by $\Pi_p(E,F)$. The norm $\pi_p(A)$ is given by the infimum of all numbers $C>0$ such that for all $\{x_i\}_{i=1}^n \subset E$, $n\in\mathbb{N}$, we have

$$\left(\sum_{i=1}^n \|A(x_i)\|^p\right)^{1/p} \le C \sup_{\|x^*\|=1} \left(\sum_{i=1}^n |\langle x_i,x^*\rangle|^p\right)^{1/p}.$$

$\Gamma_p(E,F)$ is the space of all operators that factor through a space $L^p(\mu)$: The norm is

$$\gamma_p(A)=\inf\{\|B\|\|C\| \mid A=CB,\ B\in L(E,L^p(\mu)),\ C\in L(L^p(\mu),F)\}.$$

There are several notions that are related to the unconditional structure. We say that a basis $\{x_i\}_{i\in I}$, I possibly infinite, is C-unconditional if we have for all $a_i\in\mathbb{R}$, $\varepsilon_i=\pm 1$, $i\in I$

$$\|\sum_{i\in I} a_i x_i\| \le C\|\sum_{i\in I} \varepsilon_i a_i x_i\|.$$

The infimum over all such C is denoted by $ubc(\{x_i\}_{i\in I})$, the unconditional basis constant of the basis $\{x_i\}_{i\in I}$. The unconditional basis constant of a space E is

$$ubc(E)=\inf\{ubc(\{x_i\}_{i\in I}) \mid \{x_i\}_{i\in I} \text{ is a basis of } E\}.$$

A Banach space E has local unconditional structure (lust) if there is a $C>0$ such that for all finite dimensional subspaces F of E there is a finite dimensional subspace G of E that contains F and satisfies $ubc(G) \le C$. The infimum of all these numbers C is denoted by lust(E).

There is a property that is - obviously - weaker than lust. It was used by Gordon and Lewis. We'd like to call this property GL-lust: There is a $C>0$ such that for all finite dimensional subspaces F of E there is a space G with $ubc(G)\le C$ and operators $A\in L(F,G)$, $B\in L(G,E)$ such that BA is the identity on F and $\|A\|\|B\| \le 1$.

Certainly, it is not a great surprise that we put GL-lust(E)=inf C.

The invariant that we shall actually compute is the Gordon-Lewis constant

$$gl(E)=\sup\{\gamma_1(A)\mid A\in\Pi_1(E,\ell^2),\ \pi_1(A)=1\}.$$

If $\{x_i\}_{i=1}^n$ is a basis of a space we denote by $\{x_i^*\}_{i=1}^n$ the biorthogonal functionals.

Quite often we consider (natural) identities. If we have such an identity, say from a space E to F, we might denote it by $id_{E,F}$ or some other way making clear between which spaces it is operating.

The ε-tensor product of two finite dimensional Banach spaces $E\otimes_\varepsilon F$ is the space of operators $L(E^*,F)$.

The dual space is the space of nuclear operators from E to F^*, $E^*\otimes_\pi F^*$.

Several tools we use, e.g. the Khintchine-inequality, can be found in [9],[12].

1. Some elementary inequalities

LEMMA 1.1 *Let E be a Banach space. Then we have*

$$gl(E)\le GL\text{-}lust(E)\le lust(E)\le ubc(E).$$

The first inequality was proved in [4]. The other inequalities follow immediately from the definitions. Pisier observed [11]

LEMMA 1.2 *Let E be a Banach space. Then we have*

$$GL\text{-}lust(E) = GL\text{-}lust(E^*).$$

LEMMA 1.3 [4] *Let E and F be Banach spaces.*

(i) $gl(F)\le d(E,F)gl(E)$

(ii) Suppose $P\in L(E,F)$ is a projection from E onto F. Then

$$gl(F)\le \|P\|\, gl(E).$$

By using Lemma 1.3 we estimate the Gordon-Lewis constant from below. In fact, we consider finite dimensional, uniformly complemented subspaces and compute the Gordon-Lewis constant for these spaces.

Since the Gordon-Lewis constant is a quotient of the γ_1- and π_1-norms of certain operators we have to estimate these norms. It turns out that it is technically easier to estimate π_1-norms. Therefore we would like to reduce the problem to estimate π_1-norms only. This is done by duality. First, we have $\gamma_1(A)=\gamma_\infty(A^t)$. Moreover, since we have for finite dimensional Banachspaces $\Pi_1(E,F)=\Gamma_\infty(F,E)^*$ [5] we have

$$\gamma_1(A)=\sup\{\frac{tr(A^tB)}{\pi_1(B)}\ |B\varepsilon L(E^*,\ell^2)\}$$

provided E is finite dimensional. Thus we obtain for finite dimensional Banach spaces E

$$(1.1) \qquad gl(E)=\sup\{\frac{tr(A^tB)}{\pi_1(A)\pi_1(B)}\ |\ A\varepsilon L(E,\ell^2),\ B\varepsilon L(E^*,\ell^2)\}.$$

The next inequality is due to Figiel and Johnson [2].

<u>*LEMMA 1.4*</u> *There is an a, $o<a<1/2$, such that for every n-dimensional Banach space E we have*

$$gl(E)\geq\sup\{\frac{a\sqrt{n}}{\sqrt{2\pi}\pi_1(A)}\inf\{\|A|_F\|\ |F\subset E,\ \dim F\geq an\}\}$$

where the supremum is taken over all invertible operators $A\varepsilon L(E,\ell^2)$.

2. Estimations of the Gordon-Lewis constant

We know that in general gl(E) is less than ubc(E). Most Banach spaces of operators have (under certain assumptions) the property that gl(E) and ubc(E) are equivalent - and big. The next theorem is taken from [15]. The proof was simplified by Szarek [17].

<u>*THEOREM 2.1*</u> *Let $\{x_i\}_{i=1}^n$ be a basis of a Banach space E. Assume that there exist constants M and K and a set G of n-tuples of signs Θ so that*

$$(2.1) \qquad \|\sum_{i=1}^n \Theta_i a_i x_i\| \leq K\|\sum_{i=1}^n a_i x_i\| \quad \text{for all scalars } \{a_i\}_{i=1}^n \text{ and all } \Theta\varepsilon G$$

$$(2.2) \qquad |G|^{-1}\sum_{\Theta\varepsilon G}|\sum_{i=1}^n \Theta_i\lambda_i| \geq M^{-1}(\sum_{i=1}^n|\lambda_i|^2)^{1/2} \quad \text{for all scalars } \{\lambda_i\}_{i=1}^n .$$

Then we have

$$ubc(\{x_i\}_{i=1}^n) \leq K^2M^2 gl(E).$$

For the proof we require a lemma.

<u>*LEMMA 2.2*</u> *Let $\{x_i\}_{i=1}^n$ be a basis of a Banach space E and suppose that (2.1) and (2.2) are valid. Then we have for all diagonal operators $T\varepsilon L(E^*,\ell_n^2)$, $T(\sum_{i=1}^n a_i x_i^*)=(|t_i|a_i)_{i=1}^n$,*

$$\pi_1(T) \leq KM\|\sum_{i=1}^n t_i x_i\|.$$

<u>Proof.</u> By (2.2) we have for $y_\ell=\sum_{i=1}^n \eta_i^\ell x_i^*$, $\ell=1,\dots\dots,N$

$$M^{-1}\sum_{\ell=1}^{N}\|T(y_\ell)\| = M^{-1}\sum_{\ell=1}^{N}\Big(\sum_{i=1}^{n}|\eta_i^\ell t_i|^2\Big)^{1/2}$$
$$\leq |G|^{-1}\sum_{\ell=1}^{N}\sum_{\Theta\in G}\Big|\sum_{i=1}^{n}\Theta_i\eta_i^\ell t_i\Big|$$
$$\leq \max_{\Theta\in G}\sum_{\ell=1}^{N}\Big|\sum_{i=1}^{n}\Theta_i\eta_i^\ell t_i\Big| .$$

Now we shall apply (2.1). We obtain

$$M^{-1}\sum_{\ell=1}^{N}\|T(y_\ell)\| \leq \max_{\Theta\in G}\max_{\|x\|=1}\Big\|\sum_{i=1}^{n}\Theta_i t_i x_i\Big\|\sum_{\ell=1}^{N}\Big|\Big\langle\sum_{i=1}^{n}\eta_i^\ell x_i^*,x\Big\rangle\Big|$$
$$\leq K\Big\|\sum_{i=1}^{n}t_i x_i\Big\|\max_{\|x\|=1}\sum_{\ell=1}^{N}|\langle y_\ell,x\rangle| . \qquad \square$$

<u>Proof of Theorem 2.1</u> Consider diagonal operators $S\in L(E,\ell_n^2)$, $T\in L(E^*,\ell_n^2)$ as in Lemma 2.2. By (1.1) and Lemma 2.2 we obtain

$$gl(E)=\sup\Big\{\frac{tr(S^tT)}{\pi_1(S)\pi_1(T)}\ \Big|\ S\in L(E,\ell_n^2),\ T\in L(E^*,\ell_n^2)\Big\}$$
$$\geq K^{-2}M^{-2}\sup\Big\{\frac{\sum_{i=1}^{n}|s_i t_i|}{\|\sum_{i=1}^{n}s_i x_i^*\|\ \|\sum_{i=1}^{n}t_i x_i\|}\ \Big|\ \{s_i\}_{i=1}^n\ ,\ \{t_i\}_{i=1}^n\Big\}.$$

Therefore we get

$$K^2M^2gl(E) \geq ubc(\{x_i\}_{i=1}^n) . \qquad \square$$

We say that a double sequence $\{x_{ij}\}_{i,j=1}^{n,m}$ is C-tensor-unconditional if we have for all $\varepsilon_i=\pm1$, $\delta_j=\pm1$, $a_{ij}\in\mathbb{R}$, $i=1,\dots,n$, $j=1,\dots,m$

$$\Big\|\sum_{i,j=1}^{n,m}a_{ij}x_{ij}\Big\| \leq C\Big\|\sum_{i,j=1}^{n,m}\varepsilon_i\delta_j a_{ij}x_{ij}\Big\| .$$

This notion and the following corollary applies in particular to spaces of operators.

<u>*COROLLARY 2.3*</u> *Let E be a Banach space with a basis* $\{x_{ij}\}_{i,j=1}^{n,m}$ *that is 1-tensor-unconditional. Then we have*

$$(2.3)\qquad ubc(\{x_{ij}\}_{i,j=1}^{n,m}) \leq 4gl(E) .$$

<u>Proof.</u> As a basis in Theorem 2.1 we choose $\{x_{ij}\}_{i,j=1}^{n,m}$ and $G = \{(\varepsilon_i\delta_j)_{i,j=1}^{n,m}\,|\,\varepsilon_i=\pm1,\ \delta_j=\pm1\}$. Therefore we have K = 1. By using the

Khintchine-inequality [9] twice and the triangle-inequality once we get (2.2) with M = 2. □

REMARK. (i) V. Samarsky [14] proved independently a weaker version of this corollary.
(ii) In inequality (2.3) appears 4 as a constant. We do not know what the best possible constant is. Nevertheless, we know that it has to be greater than or equal to 2. Indeed, consider $\ell_2^1 \otimes_\varepsilon \ell_2^1$ and choose $x_{ij}=e_i \otimes e_j$, i,j=1,2. We obtain

$$\mathrm{ubc}(\{e_i \otimes e_j\}_{i,j=1}^2) \geq \frac{\left\|\begin{pmatrix}1&1\\1&1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}1&1\\1&-1\end{pmatrix}\right\|} = \frac{4}{2} = 2.$$

On the other hand, $d(\ell_2^1, \ell_2^\infty)=1$ and therefore $1=d(\ell_2^1 \otimes_\varepsilon \ell_2^1, \ell_2^\infty \otimes_\varepsilon \ell_2^\infty) = d(\ell_2^1 \otimes_\varepsilon \ell_2^1, \ell_4^\infty)$. But, $\mathrm{gl}(\ell_4^\infty)=1$.

3. Unitary operator ideals

As in the whole paper we also restrict ourselves here to the real case. We consider the space of finite rank operators $A \in L(H)$ on a possibly finite dimensional Hilbert space H. We call a norm α_E on this space unitarily invariant if $\alpha_E(UAV)=\alpha_E(A)$ where U and V are isometries of H and if $\|A\| \leq \alpha_E(A) \leq n(A)$, i.e. $\alpha_E(A)$ is larger than the operator norm and less than the nuclear norm. The completion of this space under α_E is a unitary operator ideal $C_E(H)$.
Consider now $H=\ell^2$ or ℓ_n^2 and a diagonal operator $T((a_i)_{i\in I})=(t_i a_i)_{i\in I}$. Then there is a space E with a symmetric basis $\{e_i\}_{i\in I}$ (i.e. for all permutations π and all $a_i \in \mathbb{R}$ we have $\|\Sigma a_i e_i\| = \|\Sigma \pm a_{\pi(i)} e_i\|$) such that

$$\alpha_E(T) = \left\| \sum_{i\in I} t_i e_i \right\| .$$

By polar decomposition the computation for the norm of an operator can be reduced to this case. Therefore one also says that the symmetric space E "generates" the operator ideal C_E.
If $E=\ell^2$ we have the space of Hilbert-Schmidt operators which we prefer to denote by HS and its norm by hs( ). Also, if $E=\ell^p$, $p\neq\infty$, we write C_p.
It was shown by D.R.Lewis [8] that C_E has local unconditional structure if and only if E is isomorphic to a Hilbert space. This result was refined in [10].

THEOREM 3.1 *There is a $C>0$ such that we have for all $C_E(\ell_n^2)$, $n\in\mathbb{N}$,*

$$gl(C_E)\le lust(C_E)\le ubc(C_E)\le d(C_E,HS)\le C\,gl(C_E).$$

We require a proposition that is essentially due to D.R.Lewis.

PROPOSITION 3.2 *There is a $C>0$ such that we have for all $C_E(\ell_n^2)$, $n\in\mathbb{N}$*

$$\pi_1(id_{C_E,HS})\le Cn\|id_{HS,C_E}\|^{-1}$$

where $id_{C_E,HS}$ denotes the natural identity between C_E and HS.

Proof of Theorem 3.1 By (1.1) and Proposition 3.2 it follows that

$$\begin{aligned} gl(C_E) &\ge n^2[\pi_1(id_{C_E,HS})\pi_1(id_{C_E^*,HS})]^{-1} \\ &\ge C^{-2}\|id_{C_E,HS}\|\,\|id_{C_E^*,HS}\| \\ &= C^{-2}\|id_{C_E,HS}\|\,\|id_{HS,C_E}\|\ge C^{-2}d(C_E,HS). \end{aligned}$$

The other inequalities follow from Lemma 1.1 . □

The next lemma is a nonessential modification of a result of Helgason [7] and Figa-Talamanca and Rider [1] . O_n denotes the orthogonal group on ℓ_n^2 and σ its normalized Haar measure.

LEMMA 3.3 [10] *There is a constant $C>0$ such that we have for all A, $B\in L(\ell_n^2)$, $n\in\mathbb{N}$,*

$$\int_{O_n}\int_{O_n}|tr(AUBV)|\,d\sigma(U)\,d\sigma(V)\ \ge\ C\frac{hs(A)hs(B)}{n}\ .$$

Proof of Proposition 3.2

$$\begin{aligned} \sup_{\alpha_{E^*}(B)=1}\sum_{\ell=1}^{N}|tr(A_\ell B)| &\ge \sup_{\alpha_{E^*}(B)=1}\sum_{\ell=1}^{N}\int\int_{O_n}|tr(A_\ell UBV)|\,d\sigma(U)\,d\sigma(V) \\ &\ge \sup_{\alpha_{E^*}(B)=1} C\sum_{\ell=1}^{N}\frac{hs(A_\ell)hs(B)}{n} \\ &= Cn^{-1}\|id_{C_{E^*},HS}\|\sum_{\ell=1}^{N}hs(A_\ell). \end{aligned}$$

□

COROLLARY 3.4 [8] *$C_E=C_E(\ell^2)$ has lust if and only if E is isomorphic* *to* ℓ^2.

This corollary follows immediately by reduction to the finite dimensional case. There are projections of norm 1 from $C_E(\ell^2)$ onto $C_E(\ell^2_n)$. So, we just apply Lemma 1.3 and Theorem 3.1 .
By this it is also clear that there are 1-absolutely summing operators that do not factor through a space $L^1(\mu)$. This is done by putting finite rank operators together.

COROLLARY 3.5 *There are operators from* $L(\ell^2)$ *to* HS, *the space of Hilbert-Schmidt operators, that are one absolutely summing but do not factor through a space* $L^1(\mu)$.

4. ε-tensor product

PROPOSITION 4.1 [16] *Suppose* $\{e_i\}_{i=1}^n$ *and* $\{f_j\}_{j=1}^m$ *are 1-unconditional bases of the Banach spaces* E *and* F, *respectively. Then*

$$C(\log(n+m))^{-1/2}\min\{d(E,\ell^\infty_n)^{1/2}, d(F,\ell^\infty_m)^{1/2}\}$$

$$\leq gl(E\otimes_\varepsilon F)\leq ubc(E\otimes_\varepsilon F) \tag{4.1}$$

$$\leq \min\{d(E,\ell^\infty_n), d(F,\ell^\infty_m)\}$$

where $C>0$ *is an absolute constant.*

REMARK. This result was refined by Gordon [3]. He showed essentially that assumptions on ubc(E) and ubc(F) are not necessary.

Thus the question was raised whether the log-term in the left hand side of (4.1) is necessary. As a matter of fact, the log-term entered because one passed from Gaussian to Bernoulli random variables. Since it turned out that the log-term is essential it also reflects the difference between Gaussian and Bernoulli random variables.

THEOREM 4.2 [16] *There is a constant* $C>0$ *and a sequence of spaces* E_n, $n\in\mathbb{N}$, $ubc(E_n)=1$, *such that* $d(E_n,\ell^\infty_{\dim E_n})$ *tends with n to infinity and such that we have for all* $n\in\mathbb{N}$

$$ubc(E_n\otimes_\varepsilon E_n)\leq C.$$

We give the construction of the spaces E_n. E_n is the space $\mathbb{R}^n$ equipped with a norm defined for the vectors $\sum_{i=1}^{k} e_i$, where e_i, $i=1,\dots,n$ are the natural unit vectors

$$k_1=1,\quad k_{j+1}=2^{k_j},\quad j=1,2,3,\dots\dots$$

$$\|\sum_{i=1}^{k} e_i\| = \phi(k)=j \quad \text{for } k_j \le k < k_{j+1}.$$

For the dual basis $\{e_i^*\}_{i=1}^n$ and all permutations π of $\{1,\dots\dots,n\}$ we put

$$\|\sum_{i=1}^{k} e^*_{\pi(i)}\| = k\|\sum_{i=1}^{k} e_i\|^{-1}.$$

Now we take the convex hull of

$$M=\{\|\sum_{i=1}^{k} e_i^*\|^{-1} \sum_{i=1}^{k} \pm e^*_{\pi(i)} \mid k=1,\dots,n,\ \pi \text{ is permutation of } \{1,\dots,n\}\}$$

as the dual unit ball. Please note that $\{e_i\}_{i=1}^n$ is a 1-unconditional basis. In order to show $ubc(E_n \otimes_\varepsilon E_n) \le C$ we use a solution of Zarankiewicz's problem [13] and the following property of the spaces E_n. Consider the function $r:\{8^8, 8^8+1,\dots\dots\} \to \mathbb{N}$

$$r(k)=[\log_2\log_2\log_2\log_2 k].$$

We observe that there is a $C>0$ such that for all $k=8^8, 8^8+1,\dots.$ we have

$$\|\sum_{i=1}^{k} e_i\| \le C\,\|\sum_{i=1}^{r(k)} e_i\|.$$

5. Unconditional structure of subspaces of $\ell_n^2 \otimes_\varepsilon \ell_n^2$

So far we found out that the Gordon-Lewis constant of $\ell_n^2 \otimes_\varepsilon \ell_n^2$ is of the order $\sqrt{n}$. But we did not obtain anything for subspaces. Clearly, $\ell_n^2 \otimes_\varepsilon \ell_n^2$ contains ℓ_n^2 as a subspace and for ℓ_n^2 we have certainly $gl(\ell_n^2)$ $=ubc(\ell_n^2)=1$. But what if we consider a subspace of larger dimension? We get that subspaces of $\ell_n^2 \otimes_\varepsilon \ell_n^2$ with sufficiently big dimension have big Gordon-Lewis constants. We restrict ourselves here to the case $\ell_n^2 \otimes_\varepsilon \ell_n^2$. But the same can be done for various spaces.
The main idea used here is due to Figiel and Johnson [2]. They considered a different class of spaces. The result presented here can be found in [10].

<u>*THEOREM 5.1*</u> *There is a constant $C>0$ such that for all k-dimensional subspaces E of $\ell_n^2 \otimes_\varepsilon \ell_n^2$, $n,k \in \mathbb{N}$, we have*

$$Ckn^{-3/2} \le gl(E).$$

If k is of the order of n this estimate is certainly optimal. But for other k this is not known.

LEMMA 5.2 [10] *If E is an k-dimensional subspace of $\ell_n^2 \otimes_\varepsilon \ell_n^2$ then we have for the natural identity $id \in L(E,HS)$*

$$\|id\| \ge \sqrt{\frac{k}{2n}}.$$

Now, Theorem 5.1 follows from Lemma 1.4 and 5.2 and Proposition 3.2.

6. Embedding spaces without GL-lust into spaces with GL-lust

It is clear that one cannot embed a space without GL-lust complementably into a space with GL-lust. The assumption of complementation is here essential. Indeed, since every Banach space can be found as a subspace of a space $L^\infty(\mu)$.

Nevertheless, Pisier proved [11] that one cannot find certain spaces without GL-lust as subspaces in a space with lust if one assumes that the space does not contain ℓ_n^∞'s uniformly (we say that E contains ℓ_n^∞'s uniformly if there is a C>0 and subspaces $E_n \subset E$ such that for all $n \in \mathbb{N}$ we have $d(E_n, \ell_n^\infty) \le C$).

This applies in particular to the spaces C_p, $p \neq 2$.

We would like to introduce the invariant "matrix-average" of a space E. Let C>0 be so that for all double sequences $\{x_{ij}\}_{i,j=1}^n$, all $\varepsilon_{ij} = \pm 1$, i,j=1,.....,n, $n \in \mathbb{N}$, we have

$$(6.1) \qquad 2^{-2n} \sum_{\delta,\eta} \|\sum_{i,j=1}^n \varepsilon_{ij}\delta_i\eta_j x_{ij}\|^2 \le C^2 2^{-2n} \sum_{\delta,\eta} \|\sum_{i,j=1}^n \delta_i\eta_j x_{ij}\|^2$$

where δ and η range over all possible sequences of ± 1's. We denote the infimum of all these C by mave(E).

THEOREM 6.1 [11] *Let E be an infinite dimensional Banach space with GL-lust(E)<∞. Moreover, suppose that E does not contain ℓ_n^∞'s uniformly. Then we have*

$$mave(E) < \infty.$$

COROLLARY 6.2 *The Banach spaces C_p, $p \neq 2$, are not isomorphic to a subspace of a Banach lattice that does not contain ℓ_n^∞'s uniformly.*

In particular, C_p is not isomorphic to a subspace of a uniformly convex Banach lattice.

To show the first statement of the corollary it is sufficient to show that $\text{mave}(C_p)=\infty$. Let $\{x_i\}_{i=1}^{\infty}$ be an orthonormal basis in ℓ^2. Then we have for all $a_{ij}\in\mathbb{R}$, δ_i, $\eta_j=\pm1$, $i,j=1,\ldots\ldots,n$

$$\|\sum_{i,j=1}^{n} a_{ij}x_i\otimes x_j\|_{C_p} = \|\sum_{i,j=1}^{n} \delta_i\eta_j a_{ij}x_i\otimes x_j\|_{C_p}.$$

Therefore, if $\text{mave}(C_p)\le C$, (6.1) means

$$\|\sum_{i,j=1}^{n} \varepsilon_{ij}a_{ij}x_i\otimes x_j\|_{C_p} \le C\|\sum_{i,j=1}^{n} a_{ij}x_i\otimes x_j\|_{C_p}$$

for all $\varepsilon_{ij}=\pm1$. In other words, $\text{ubc}(\{x_i\otimes x_j\}_{i,j=1}^{n}) \le C$ for all $n\in\mathbb{N}$. But, by Theorem 3.1 this is not possible.
To verify the second statement of the corollary it is left to see that a uniformly convex Banach space does not contain ℓ_n^{∞}'s uniformly.

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DUALITY AND GEOMETRY OF SPACES OF COMPACT OPERATORS

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INTRODUCTION

The object of this paper is to discuss the problem of how - for given Banach spaces X and Y - the space K(X,Y) of compact linear operators from X into Y and its dual space reflect the geometric and topological properties of X and Y and their respective duals.

Since X^* and Y are closed linear subspaces of K(X,Y), they inherit trivially properties such as non-containment of l_1, the dual having the Radon-Nikodym property, weak sequential completeness, or reflexivity, from K(X,Y). The natural question thus is to find out which additional conditions are needed for the reverse implications: how can geometric and topological properties of K(X,Y) and its dual be recovered from the corresponding properties of (the presumably well known spaces) X and Y and their duals?

Besides the properties already mentioned, we shall consider the representation of the dual of K(X,Y), and discuss how the various classes of extreme points of the dual unit ball of K(X,Y) can be represented by the corresponding classes of points in the (bi-) duals of X and Y.

The paper is essentially written in survey form, but, except for section 5, the main stream results are presented with proofs.

The spectrum of results considered here will cover only a very small sector of the subject matter and is restricted to those parts that I have been involved with during the past few years.

1. PRELIMINARIES

1.1 SPACES OF COMPACT OPERATORS

Our discussion is placed in the general context of the operator space $K_{w^*}(X^*,Y)$ of compact and weak*-weakly continuous linear operators from X^* into Y, endowed with the usual operator norm. This space, originally introduced by L. Schwartz [60] in 1957 as the so-called ε-product $X\varepsilon Y$ of X and Y, apparently has gone out of sight over the years. However, it has the advantage over K(X,Y) that, as far as methods of proofs are concerned, it is conceptually easier to handle than K(X,Y) itself, and that it comprises not only spaces of the type K(X,Y) but also completed injective tensor products $X\tilde{\otimes}_\varepsilon Y$, and thus many more concrete spaces of analysis - particularly spaces of vector-valued continuous functions and of vector-valued measures. More specifically, we have the following (well known) fundamental isometrical isomorphisms and isometrical embeddings, respectively:

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*) This paper is based on joint work with H.S. Collins, Baton Rouge (Louisiana) [4,5] and C.P. Stegall, Linz (Österreich) [52,53].

(1.1) $K(X,Y) = K_{w^*}(X^{**},Y)$
$k \longmapsto k^{**}$

(1.2) $X \tilde{\otimes}_\varepsilon Y \hookrightarrow K_{w^*}(X^*,Y)$
$x\otimes y \longmapsto \{x^* \longmapsto (x^*x)y\}$, and $X \tilde{\otimes}_\varepsilon Y = K_{w^*}(X^*,Y)$ if

X or Y has the approximation property, cf. [60].

(1.2.a) $C(K,X) = K_{w^*}(X^*,C(K)) = C(K) \tilde{\otimes}_\varepsilon X$
$F \longmapsto \{x^* \longmapsto x^*F\}$ (K compact Hausdorff)

(1.2.b) $cca(\Sigma,X) = K_{w^*}(X^*,ca(\Sigma)) = ca(\Sigma) \tilde{\otimes}_\varepsilon X$
$\Phi \longmapsto \{x^* \longmapsto x^*\Phi\}$

(Here, Σ is a σ-algebra (of subsets of a nonempty set Ω), and $cca(\Sigma,X)$ denotes the space of countably additive measures from Σ into X with relatively compact range, endowed with the semi-variation norm, cf.[18,51].)

Duality results for $K_{w^*}(X^*,Y)$ thus often allow immediate specializations to both K(X,Y)-spaces and $X \tilde{\otimes}_\varepsilon Y$-spaces, and this is the general approach that we use in this paper.

1.2 NOTATION

Basically, we follow the classical notation of Dunford and Schwartz [12]. A subset A of a Banach space X is called *weakly sequentially precompact* if every sequence in A has a weak Cauchy subsequence. The unit ball of a Banach space X is denoted by B_X, and its extreme points by $ext B_X$.

The spaces of all bounded, weakly compact, compact, and nuclear operators from X into Y - X and Y Banach spaces - are denoted by L(X,Y), W(X,Y), K(X,Y), and N(X,Y), respectively. In accordance with our definition of $K_{w^*}(X^*,Y)$, we denote by $L_{w^*}(X^*,Y)$ the space of all weak*-weakly continuous linear operators from X^* into Y.

The term *Radon-Nikodym property* is, as usual, abbreviated by *RNP*, and the term [*metric*] *approximation property* by [*m.*]*a.p.*

2. EARLY RESULTS

We briefly recall several classical results on the duality and geometry of K(X,Y) in order to give a flavour of the results to be expected - and those not to be expected.

2.1 Dunford / Schatten [11], 1946, and Schatten [58], 1950:
For $1 < p < \infty$, *the space* $K(l_p)$ *contains a copy of* c_0 *isometrically.*

2.2 Schatten [59], 1957: *The unit ball of* $K(l_2)$ *has no extreme points.*

2.3 Corollary (Dunford / Schatten): *Let H be an infinite-dimensional Hilbert space. Then K(H) is not reflexive, and not even weakly sequentially complete, nor isometric to a dual space.*

These are, in a sense, the negative results. The message is that one cannot expect K(X,Y) to be "nice" provided that only X and Y are "nice" enough. On the contrary, K(X,Y) may just become too large.

In the other direction, there are the following classical positive results.

2.4 Pitt [45], 1936: *For* $1 \leq p < q < \infty$, *all bounded linear operators from* l_q *into* l_p *are compact:* $L(l_q, l_p) = K(l_q, l_p)$.

Combined with Grothendieck's result 2.6(b)(i) quoted below, this implies that *for* $1 \leq p < q < \infty$, *the space* $K(l_q, l_p)$ *is reflexive.*

2.5 Grothendieck [19], 1955: *Consider the canonical (linear) map*

$$j : X^* \tilde{\otimes}_\pi Y^* \longrightarrow (K_{w^*}(X^*, Y))^*$$
$$x^* \otimes y^* \longmapsto \{h \longmapsto (hx^*, y^*)\}.$$

Then we have:

(a) j is surjective if X^* *or* Y^* *has RNP.*

(b) j is injective if X^* *or* Y^* *has a.p.*

More specifically: If either of X^* *and* Y^* *has a.p., then the map*

$$p : X^* \tilde{\otimes}_\pi Y^* \longrightarrow B(X,Y)$$
$$x^* \otimes y^* \longmapsto \{(x,y) \longmapsto (x^*x)(y^*y)\}$$

is injective, and if p is injective, then so is j.

2.6 Corollary (Grothendieck [19], 1955):

(a) If X^{**} *or* Y^* *has RNP, then* $K(X,Y)^*$ *is a quotient of* $X^{**} \tilde{\otimes}_\pi Y^*$, *and if, in addition, either of* X^{**} *and* Y^* *has a.p., then we have:* $K(X,Y)^* = X^{**} \tilde{\otimes}_\pi Y^* = N(X^*, Y^*)$.

(b) Assume that X and Y are reflexive. Then we have:

(i) If $L(X,Y) = K(X,Y)$, *then* $K(X,Y)$ *is reflexive.*

(ii) Conversely, if $K(X,Y)$ *is reflexive and* $p : X^{**} \tilde{\otimes}_\pi Y^* \longrightarrow B(X^*, Y)$ *is injective, then* $L(X,Y) = K(X,Y)$.

We pause for a proof of Corollary 2.6, for these consequences of Grothendieck's results 2.5 later on have often been reconsidered (see Remarks 2.7 below).

Proof of Corollary 2.6: Proposition (a) is - via the isometry (1.1) - just a special case of 2.5. For a proof of proposition (b), assume now that X and Y are reflexive. Then, according to (a), we have $K(X,Y)^* = X \tilde{\otimes}_\pi Y^*/Q$, so that $K(X,Y)^{**} = Q^\perp \subset L(X,Y)$. A quick inspection of the corresponding isometrical embeddings reveals that $K(X,Y)$ is reflexive in case $L(X,Y) = K(X,Y)$. This proves proposition (b)(i). As for (b)(ii), we need only use 2.5 to conclude from $K(X,Y)^* = X \tilde{\otimes}_\pi Y^*$ (and the assumption) that $K(X,Y) = K(X,Y)^{**} = (X \tilde{\otimes}_\pi Y^*)^* = L(X,Y)$.

2.7 Remarks: 1. Theorem 2.5 is possibly better known as the result on the coincidence of integral and nuclear operators in the presence of RNP. Grothendieck [19,I.§4.2,Thm.8] did not explicitly state 2.5, but proved it for $X \tilde{\otimes}_\varepsilon Y$ under the assumption that X is reflexive or has a separable dual, and that $p : X^* \tilde{\otimes}_\pi Y^* \longrightarrow B(X,Y)$ is injective. In his proof, however, he only uses that, under the assumptions on X, the dual unit ball B_{X^*} has what he called the "Phillips property" ("propriêtê Φ") [19,I.§4.1,Def.6,p.104] which, in present day language, exactly is the RNP for X^*.

For an explicit proof of 2.5, see Diestel [7], Gil de Lamadrid [16], Schachermayer [57], and C. Stegall [66,Cor.7].

2. The results of Corollary 2.6, only implicit in Grothendieck's thesis [19] - as indicated above - have explicitly been proved by various authors. Feder and Saphar [14,Thm.1] proved proposition (a), also relying heavily on Grothendieck's work [19]. Proposition (b) appears in various (partly weaker) versions in Ruckle [50], Holub [28,29], Kunas K. Jun [34], Kalton [35,section 2,Cor.2], and, in the form stated above, in Heinrich [22].

3. Note that proposition 2.6(b)(ii) particularly implies Dunford / Schatten's result that, for an infinite-dimensional Hilbert space H, K(H) is not reflexive.

Historically, looking back, it is not too surprising that after the early results on spaces of operators on Banach spaces mentioned so far, not much happened in this direction for about a decade, and that it took the vivid revival of Banach space theory in the sixties to renew the interest in the geometric and topological structure also of operator spaces. This process did not start before the early seventies. A survey on a small part of it is the subject of the subsequent sections.

Note: It seems worth specifying the assertion of Theorem 2.5 in the sense of our general program - to determine K(X,Y) and its dual from the spaces X and Y and their duals:

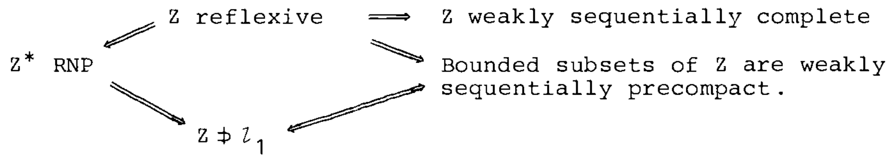
(a) If either of X^{**} and Y^* has RNP, then every $T \in K(X,Y)^*$ has a representation of the form $T = \Sigma\lambda_i x_i^{**} \otimes y_i^*$, with $(\lambda_i)_i \in l_1$, and $(x_i^{**})_i$ and $(y_i^*)_i$ bounded sequences in X^{**} and Y^*, respectively, in the sense that $Tk = \Sigma\lambda_i (k^{**}x_i^{**}, y_i^*)$ for all $k \in K(X,Y)$.

(b) If X^* has RNP, then every $\Phi \in C(K,X)^*$ (K compact Hausdorff) has a representation of the form $\Phi = \Sigma\lambda_i x_i^* \otimes \mu_i$, with $(\lambda_i)_i \in l_1$, and $(x_i^*)_i$ and $(\mu_i)_i$ bounded sequences in X^* and M(K), respectively, in the sense that $\Phi F = \Sigma\lambda_i \int x_i^* F d\mu_i$ for all $F \in C(K,X)$.

This is to be noted as the advantage over the case for general X, where the dual of C(K,X) can only be represented as a space of more general X^*-valued measures, cf. Singer [63], Prolla [46,Ch.V.5] and Schmets [61,p.368-377].

3. REFLEXIVITY AND WEAKER NOTIONS

We first recall the implications among the various weakenings of reflexivity.



Any of these properties is being inherited by X and Y from the operator space $K_{w*}(X^*,Y)$. According to our general theme, we now turn to the discussion of which additional assumptions are required for the reverse implications.

3.1 REFLEXIVITY AND WEAK SEQUENTIAL COMPLETENESS

Generally speaking (and modulo the a.p.), for K(X,Y) to be reflexive or just weakly sequentially complete, all bounded operators from X into Y must necessarily be compact. More specifically, the following results hold.

3.1.1 Grothendieck [19] (see 2.5 and 2.6 above):
Assume that X and Y are reflexive and that either of X and Y has a.p. Then K(X,Y) is reflexive if and only if L(X,Y) = K(X,Y).

3.1.2 Theorem (D.R. Lewis [37,Thm.2.1]):
(a) Assume that X and Y are weakly sequentially complete, and that $L_{w*}(X^*,Y) = K_{w*}(X^*,Y)$. *Then* $K_{w*}(X^*,Y)$ *is weakly sequentially complete.*
(b) Conversely, assume that $K_{w*}(X^*,Y)$ *is weakly sequentially complete, and that either of X and Y has the metric approximation property. Then X and Y are weakly sequentially complete, and* $L_{w*}(X^*,Y) = K_{w*}(X^*,Y)$.

Notes: Proposition (a) as stated here is to be found in [4,Thm.3.4]. D.R. Lewis [37,Thm.2.1] proved it for the space $X \tilde{\otimes}_\varepsilon Y$ and thus needed the a.p. in one of the factors also for this part of the assertion.
A special case of proposition (a) is the result of F. Lust [40].

What is behind Theorem 3.1.2 is the following general result.

3.1.2' [4,Prop.3.1]: $K_{w*}(X^*,Y)$ *is weakly sequentially complete if and only if*
(i) X and Y are weakly sequentially complete, and
(ii) $K_{w*}(X^*,Y)$ *is weak-operator-topology sequentially closed in* $L_{w*}(X^*,Y)$.

Thus, D.R. Lewis' particular achievement - with his beautiful proof of proposition 3.1.2(b) in [37] - was to show that condition (ii) in 3.1.2' together with the m.a.p. in one of the factors implies equality of $L_{w*}(X^*,Y)$ and $K_{w*}(X^*,Y)$.

3.1.3 Corollary (D.R. Lewis [37,Cor.2.4], and [4,Thm.3.5]): *(a) Assume that X^* and Y are weakly sequentially complete and that $W(X,Y) = K(X,Y)$. Then $K(X,Y)$ is weakly sequentially complete.*
(b) Conversely, assume that $K(X,Y)$ is weakly sequentially complete and that either of X^* and Y has m.a.p. Then X^* and Y are weakly sequentially complete and $W(X,Y) = K(X,Y)$.

This is just a special case of 3.1.2 by using the isometries $K(X,Y) = K_{w^*}(X^{**},Y)$ and $W(X,Y) = L_{w^*}(X^{**},Y)$.

For further special cases (and extensions to particular locally convex spaces), the reader is referred to section 3 of [4], and to Tsitsas [70,71].

Note, finally, that 3.1.3 implies Dunford / Schatten's result that $K(H)$ is not weakly sequentially complete if H is an infinite-dimensional Hilbert space.

3.2 THE DUAL HAVING THE RADON-NIKODYM PROPERTY

In contrast with the situation for reflexivity, weak sequential completeness and non-containment of l_1 (3.3 below) - where additional assumptions are needed - the RNP for both X^* and Y^* completely determines the RNP for the dual of $K_{w^*}(X^*,Y)$.

3.2.1 Theorem [52,Thm.1.9]: *The dual of $K_{w^*}(X^*,Y)$ has RNP if and only if X^* and Y^* have RNP.*

3.2.2 Corollary [52,Cor.1.10]: *The dual of $K(X,Y)$ has RNP if and only if X^{**} and Y^* have RNP.*

A particular consequence of Theorem 3.2.1 is the result that - in case either of X^* and Y^* has a.p. - the space $N(X,Y^*) = X^* \tilde{\otimes}_\pi Y^*$ of nuclear operators from X into Y^* has RNP, provided that X^* and Y^* have RNP; for, under these assumptions, we have $K_{w^*}(X^*,Y) = X \tilde{\otimes}_\varepsilon Y$ and $K_{w^*}(X^*,Y)^* = X^* \tilde{\otimes}_\pi Y^* = N(X,Y^*)$ (see 2.5 above). This special case of Theorem 3.2.1 is proved in Diestel / Uhl [10,VIII.4,Thm.7,p.249], and it is their method of proof that we generally follow in our proof of Theorem 3.2.1.

Proof of Theorem 3.2.1: We use the fact that Z^* has RNP if and only if every separable subspace of Z has a separable dual: Uhl, cf. [10,III.3,Cor.6,p.82] and Stegall [66]. From this equivalence, the necessity part of Theorem 3.2.1 follows trivially. Now, assume that X^* and Y^* have RNP, and let A be a closed separable subspace of $K_{w^*}(X^*,Y)$. Choose a dense sequence $(h_n)_n$ in A, and a closed separable subspace Y_o of Y such that $\operatorname{im} h_n \subset Y_o$ for all $n\in\mathbb{N}$. Accordingly, for $(h_n^*)_n \subset K_{w^*}((Y_o)^*,X)$, choose a closed separable subspace X_o of X such that $\operatorname{im} h_n^* \subset X_o$ for all $n\in\mathbb{N}$. Thus, we can view the sequence $(h_n)_n$ as a subset of $K_{w*}((X_o)^*,Y_o)$. Now, it is generally true (cf. [60]) that for M and N closed linear subspaces of X and Y, respectively, the space $K_{w*}(M^*,N)$ is (isometrically) a closed linear subspace of $K_{w*}(X^*,Y)$. This reveals that A is a closed linear subspace of $K_{w*}((X_o)^*,Y_o)$. By assumption on X and Y, both $(X_o)^*$ and $(Y_o)^*$ are separable, so that $(X_o)^* \tilde{\otimes}_\pi (Y_o)^*$ is separable as well. According to Theorem 2.5 in section 2, the dual of $K_{w*}((X_o)^*,Y_o)$ is a quotient of $(X_o)^* \tilde{\otimes}_\pi (Y_o)^*$, and thus is separable. Consequently, the dual of the closed linear subspace A of $K_{w*}((X_o)^*,Y_o)$ is separable as well. This completes the proof.

3.3 NON-CONTAINMENT OF (AN ISOMORPH OF) l_1

3.3.1 Theorem [4,Thm.1.14]: *Assume that X and Y do not contain l_1 and that either of X^* and Y^* has RNP. Then $K_{w*}(X^*,Y)$ does not contain l_1.*

As usual, the isometry $K(X,Y) = K_{w*}(X^{**},Y)$ of (1.1) above allows the following specialization to K(X,Y).

3.3.2 Corollary [4,Cor.1.12]: *Assume that X^* and Y do not contain l_1, and that X^{**} or Y^* has RNP. Then K(X,Y) does not contain l_1.*

Notes: 1. The result of Theorem 3.3.1 has been presented by Heron Collins at the 1980 Conference on "Measure Theory and its Applications" at Northern Illinois University, DeKalb, Illinois, and first appeared in our joint announcement of results of our paper [4] in the Proceedings of that conference in 1981, edited by G.A. Goldin and R. Wheeler, p.187-192. It is interesting to note that B.M. Makarov and V.G. Samarskii announced the assertion of Corollary 3.3.2 in [41,Thm.2.3], and that, in the period when our paper [4] was in print, C. Samuel's paper [54] appeared where he proves 3.3.1 for the subspace $X \tilde{\otimes}_\varepsilon Y$. This development may help to support our point - specified in section 1.1 - that the "right" space to be looked at is the space $K_{w*}(X^*,Y)$.

2. Fakhouri [13,Thm.3] proved 3.3.2 under the assumption that X^{**} is separable.

Proof of Theorem 3.3.1: We use Rosenthal's fundamental result (see Rosenthal [49] and Odell / Rosenthal [42]) that a Banach space Z does not contain (an isomorph) of l_1 if and only if the bounded subsets of Z are weakly sequentially precompact.

Assume that $X \not\supset l_1$ and that Y^* has RNP. (Notice the symmetry $K_{w*}(X^*,Y) = K_{w*}(Y^*,X)$.) Let $(h_n)_n$ be a bounded sequence in $K_{w*}(X^*,Y)$. Then there exists a separable (closed linear) subspace Y_o of Y such that $\operatorname{im} h_n \subset Y_o$ for all $n \in \mathbb{N}$. According to our assumption, $(Y_o)^*$ is separable, see Stegall [66]. Let $(y_n^*)_n$ be a dense sequence in $(Y_o)^*$. Then $(h_n^* y_1^*)_n$ is a bounded sequence in X and thus, since $X \not\supset l_1$, has a weak Cauchy subsequence $(h_{n_k}^* y_1^*)_k$. Considering now the sequence

$(h^*_{n_k} y^*_2)_k$ in X, continuing inductively, and using a diagonal process, we arrive at a subsequence $(h_{n_l})_l$ of $(h_n)_n$ such that $(h^*_{n_l} y^*_k)_l$ is weakly Cauchy in X for all $k \in \mathbb{N}$. Taking adjoints, we conclude that $(h_{n_l} x^*)_l$ is $\sigma(Y_o, (y^*_k)_k)$-Cauchy in Y_o for all $x^* \in X^*$. But, since $(y^*_k)_k$ is norm-dense in $(Y_o)^*$, on bounded subsets of Y_o, the weak topology $\sigma(Y_o, (Y_o)^*)$ of Y_o and the topology $\sigma(Y_o, (y^*_k)_k)$ coincide. Hence, $(h_{n_l} x^*)_l$ is weakly Cauchy in Y_o, and thus in Y, for all $x^* \in X^*$. At this point, we use the isometrical embedding

$$K_{w^*}(X^*, Y) \hookrightarrow C(B_{X^*} \times B_{Y^*})$$
$$h \longmapsto \{(x^*, y^*) \longmapsto (hx^*, y^*)\},$$

to conclude from the classical weak convergence result for C(K)-spaces that $(h_{n_l})_l$ is weakly Cauchy in $K_{w^*}(X^*, Y)$. This completes the proof.

Going back to Theorem 3.3.1, and inspecting its proof - where we made heavy use of Y^* having RNP - the question arises naturally whether the assumption of the RNP for either of the duals is only a technical matter that could possibly be dispensed with by just requiring X and Y not to contain l_1. Too much hope in this direction is put to rest by the following example.

<u>3.3.3 Example [52]</u>: *Let JT denote the James Tree space, cf. [39]. Then* $JT \not\supset l_1$ *[39], but* $JT \tilde{\otimes}_\varepsilon JT$ *does contain* l_1.

Proof: JT is the dual of a (separable) Banach space B such that $B^{**}/B = l_2^c$ = Hilbert space of the dimension of the continuum [39]. Using the fact that projective tensor products "respect" quotients, we conclude that $JT^* \tilde{\otimes}_\pi JT^*$ has $l_2^c \tilde{\otimes}_\pi l_2^c$ as a quotient. Now, the "diagonal" in $l_2^c \tilde{\otimes}_\pi l_2^c$ is isometric to l_1^c, which thus can be "lifted" isomorphically to $JT^* \tilde{\otimes}_\pi JT^*$, cf. Pelczynski [43,Lemma 3.1]. But $JT^* \tilde{\otimes}_\pi JT^*$ is a closed linear subspace of $(JT \tilde{\otimes}_\varepsilon JT)^*$ (see 2.5 in section 2), so that the dual of the separable space $JT \tilde{\otimes}_\varepsilon JT$ contains l_1^c. Using Pelczynski's and Hagler's results [43,20], we conclude that $JT \tilde{\otimes}_\varepsilon JT \supset l_1$.

A combination of Corollary 3.3.2 with Theorem 2.5 yields the following connection between reflexivity and weak sequential completeness of K(X,Y).

3.3.4 Theorem ([4,Thm.3.9] and [5,Cor.3.3]): *Let X and Y be reflexive, and consider the following statements:*
(a) $L(X,Y) = K(X,Y)$; (b) $L(X,Y)$ is reflexive;
(c) $K(X,Y)$ is reflexive; (d) $K(X,Y)$ is weakly sequentially complete;
Then (a) implies (b), (b) implies (c), and (c) is equivalent to (d). Finally, if the natural map $j : X\tilde{\otimes}_\pi Y^* \longrightarrow (K(X,Y))^*$ *is injective, then (d) implies (a).* j *is injective, whenever X or Y has a.p.*

For a description of general weakly sequentially precompact subsets of $K_{w^*}(X^*,Y)$ via the corresponding concepts in the factor spaces X and Y, the reader is referred to Lewis [37,Thm.3.1], and to [4,section 1].

4. WEAK COMPACTNESS IN K(X,Y), AND EXTREME POINTS IN THE DUAL OF K(X,Y)

4.1 WEAK COMPACTNESS

We start with the well-known fact that weak sequential convergence and weak compactness are determined by convergence not necessarily on all elements of the dual but just on the extreme points of the dual unit ball.

4.1.1 Rainwater [47]: *A bounded sequence $(z_n)_n$ in a Banach space Z converges weakly to $z \in Z$ if and only if $(z^*z_n)_n$ converges to z^*z for all $z^* \in extB_{Z^*}$.*

This result was the starting point for an intense study of the reduction of weak sequential convergence and weak compactness in Banach - and more general locally convex - spaces Z to the corresponding notions with respect to the weak topology $\sigma(Z,extB_{Z^*})$ on Z generated by just the extreme points of B_{Z^*}. We refer the reader to K. Floret's detailed exposition of this development in [15].

For our purpose here, it suffices to recall one of the latest results in this direction.

4.1.2 Bourgain / Talagrand [2]: *A bounded subset H of a Banach space Z is weakly relatively compact if and only if it is relatively countably compact with respect to $\sigma(Z,extB_{Z^*})$.*

Remarks: 1. Note that, in result 4.1.2, the improvement over Rainwater's result - for describing weak relative compactness - lies in the reduction to $\sigma(Z,extB_{Z^*})$-relative countable compactness as opposed to $\sigma(Z,extB_{Z^*})$-relative sequential compactness.

2. There exist by now further proofs and extensions of Theorem 4.1.2 by Khurana [36] and Tweddle [73], a further simplification of Khurana's proof by I. Namioka (personal communication), and a deduction of result 4.1.2 from Ramsey-type theorems by C. Stegall (oral communication).

Also, it ought to be noted that for convex subsets H, 4.1.2 has been known, see De Wilde [6].

According to the discussion so far, one way to characterize weak compactness in $K_{w^*}(X^*,Y)$ is to determine the form of the extreme points in the dual unit ball of $K_{w^*}(X^*,Y)$ and then to apply 4.1.1 and 4.1.2. This approach and the subsequent results of section 4.1 are to be found in Floret [15,sections 8.10-8.13], and in [4,section 2].

4.1.3 Observation: *$\mathrm{ext}B_{(K_{w^*}(X^*,Y))^*} \subset (\mathrm{ext}B_{X^*}) \otimes (\mathrm{ext}B_{Y^*})$, where $x^*\otimes y^*$ acts on $h\in K_{w^*}(X^*,Y)$ in the canonical way: $x^*\otimes y^*(h) = (hx^*,y^*)$.*

Proof: $K_{w^*}(X^*,Y)$ is a closed linear subspace of $C(B_{X^*}\times B_{Y^*})$, B_{X^*} and B_{Y^*} being endowed with their respective weak*-topologies:

$$K_{w^*}(X^*,Y) \hookrightarrow C(B_{X^*}\times B_{Y^*})$$
$$h \longmapsto \{(x^*,y^*)\longmapsto (hx^*,y^*)\}.$$

Thus, every extreme point of $B_{(K_{w^*}(X^*,Y))^*}$ is of the form $\delta_{(x^*,y^*)}|K_{w^*}(X^*,Y) = x^*\otimes y^*$ with $(x^*,y^*)\in B_{X^*}\times B_{Y^*}$. However, if we assume $x^*\otimes y^*$ to be an extreme point of $B_{(K_{w^*}(X^*,Y))^*}$, then, by the bilinearity of the map $(x^*,y^*)\longmapsto x^*\otimes y^*$, x^* and y^* necessarily must be extreme points of B_{X^*} and B_{Y^*}, respectively. This proves the assertion.

It is now easy to arrive at the following weak compactness results, cf. Floret [15,8.10-8.13], and [4, section 2].

4.1.4 Theorem:

(a) A bounded sequence $(h_n)_n$ in $K_{w^*}(X^*,Y)$ converges weakly to $h\in K_{w^*}(X^*,Y)$ if and only if $(h_nx^*,y^*)_n$ converges to (hx^*,y^*) for all $(x^*,y^*)\in \mathrm{ext}B_{X^*}\times \mathrm{ext}B_{Y^*}$.

(b) A bounded subset H of $K_{w^*}(X^*,Y)$ is weakly relatively compact if and only if it is $\mathrm{ext}B_{X^*}\times \mathrm{ext}B_{Y^*}$-weak-operator-topology relatively countably compact.

We note two particular cases.

4.1.5 Corollary:

(a) A bounded sequence $(k_n)_n$ in $K(X,Y)$ converges weakly to $k\in K(X,Y)$ if and only if $(k_n^{**}x^{**},y^*)_n$ converges to $(k^{**}x^{**},y^*)$ for all $(x^{**},y^*)\in \mathrm{ext}B_{X^{**}}\times \mathrm{ext}B_{Y^*}$.

(b) A bounded sequence $(F_n)_n$ in $C(K,X)$, K compact Hausdorff, converges weakly to $F\in C(K,X)$ if and only if $(F_n(t))_n$ converges weakly (in X) to $F(t)$ for all $t\in K$.

For related weak compactness results, compare Kalton [35,section 2], and Tsitsas [71].

After all, we can summarize that the reduction to the $\sigma(Z,\text{ext}B_{Z^*})$-topology actually led to the desired description of weak compactness properties by means of the corresponding pointwise-weak compactness properties.

Finally, we derive a further result on the dual of $K_{w^*}(X^*,Y)$ by combining observation 4.1.3 with a result of Haydon [21,Prop.3.1].

4.1.6 Theorem ([52,Thm.1.7]): *Let H be a closed linear subspace of $K_{w^*}(X^*,Y)$, containing $X\tilde{\otimes}_\varepsilon Y$. Assume that H does not contain l_1. Then H^* is a quotient of $X^*\tilde{\otimes}_\pi Y^*$.*

Proof: The map $j : X^*\tilde{\otimes}_\pi Y^* \longrightarrow H^*$ of 2.5 in section 2 is continuous linear with $\|j\| = 1$. Since $H \not\supset l_1$, Haydon's result [21,Prop.3.1] tells us that $B_{H^*} = \text{norm-cl}\,\text{co}\,(\text{ext}B_{H^*})$. We conclude, using observation 4.1.3, that $B_{H^*} \subset \text{norm-cl}\,\text{co}\,(\text{ext}B_{X^*}\otimes\text{ext}B_{Y^*}) \subset \text{norm-cl}\,(j(B_{X^*\tilde{\otimes}_\pi Y^*})) \subset B_{H^*}$, so that $B_{H^*} = \text{norm-cl}\,(j(B_{X^*\tilde{\otimes}_\pi Y^*}))$. Banach's classical homomorphism theorem now reveals that j is a quotient map.

Example: The James Tree space JT turned out to be an example for a Banach space Z not containing l_1 but such that $Z\tilde{\otimes}_\varepsilon Z$ does contain l_1. Yet, according to Theorem 3.3.1 in section 3, $JT\tilde{\otimes}_\varepsilon JT$ ought to be very close to not contain l_1. In this sense, the James Tree space now also marks the limits of Theorem 4.1.6:

The map $j : JT^*\tilde{\otimes}_\pi JT^* \longrightarrow (JT\tilde{\otimes}_\varepsilon JT)^*$ *is <u>not</u> surjective* ([52]).

4.2 THE EXTREME POINTS OF $B_{(K_{w^*}(X^*,Y))^*}$

The observation made in section 4.1 that $\text{ext}B_{(K_{w^*}(X^*,Y))^*}$ is contained in $\text{ext}B_{X^*}\otimes\text{ext}B_{Y^*}$ naturally leads to the question whether there is actually equality between the two sets, i.e. whether it is true that for x^* and y^* extreme points in B_{X^*} and B_{Y^*}, respectively, the functional $x^*\otimes y^*$ is an extreme point of $B_{(K_{w^*}(X^*,Y))^*}$. This problem had been solved in special cases by Singer [62] for $C(K,X)$, Brosowski and Deutsch [3] and Ströbele [67] for $C_o(S,X)$, S locally compact Hausdorff, and, for general completed injective tensor products $X\tilde{\otimes}_\varepsilon Y$, by Hulanicki / Phelps [31] and Tseitlin [69]. C. Stegall and I [52] took up the general $K_{w^*}(X^*,Y)$-case and proved the following result which shows that the extreme points in the dual unit ball of the operator space $K_{w^*}(X^*,Y)$ in fact are completely determined by the extreme points in the duals of X and Y.

4.2.1 Theorem ([52,Thm.1.1]): *Let H be any linear subspace of $K_{w^*}(X^*,Y)$, containing $X\otimes Y$. Then $extB_{H^*} = (extB_{X^*})\otimes(extB_{Y^*})$.*

4.2.2 Corollary ([52,Thm.1.3]): *Let H be any linear subspace of $K(X,Y)$, containing the finite-rank operators. Then we have: $extB_{H^*} = (extB_{X^{**}})\otimes(extB_{Y^*})$.*

(For the special case of $K(l_2)$, see Holub [30,Thm.3.1].)

We established Theorem 4.2.1 in [52] by proving the following more general fact.
Given $X\otimes Y \subset H \subset K_{w^*}(X^*,Y)$, $h^*\in H^*$, $\|h^*\|=1$, and $(x_o^*,y_o^*)\in extB_{X^*}\times extB_{Y^*}$, such that $h^*|X\otimes Y = \delta_{(x_o^*,y_o^*)}|X\otimes Y$, consider the set

$C = co\{\delta_{(x_o^*,y_o^*)},-\delta_{(-x_o^*,y_o^*)},-\delta_{(x_o^*,-y_o^*)},\delta_{(-x_o^*,-y_o^*)}\}$ as a subset of $M(B_{X^*}\times B_{Y^*})$. Then we have:

(a) C is the set of norm-one extensions of h^* to $C(B_{X^*}\times B_{Y^*})$, and

(b) C is an extremal subset of $B_{M(B_{X^*}\times B_{Y^*})}$.

Here, we give a direct proof of Theorem 4.2.1 based on Tseitlin's result (thanks to S. Heinrich and W. Schachermayer).

Proof of Theorem 4.2.1: Let $(x_o^*,y_o^*)\in extB_{X^*}\times extB_{Y^*}$. Then, according to Tseitlin's result, $T = x_o^*\otimes y_o^*|X\tilde{\otimes}_\varepsilon Y \in extB_{(X\tilde{\otimes}_\varepsilon Y)^*}$. Consider the set $\Phi = \{h^*\in H^* \mid h^*|X\tilde{\otimes}_\varepsilon Y = T \text{ and } \|h^*\|=1\}$. Then Φ is a weak*-compact convex extremal subset of B_{H^*}, and thus is the weak*-closed convex hull of its extreme points. Let $h_o^*\in ext\Phi$. Then, since Φ is an extremal subset of B_{H^*}, h_o^* is an extreme point of B_{H^*}, and thus, according to observation 4.1.3 above, is of the form $h_o^* = x_1^*\otimes y_1^*$ with x_1^* and y_1^* extreme points in B_{X^*} and B_{Y^*}, respectively. Since $h_o^*\in\Phi$, we have: $x_1^*\otimes y_1^*|X\otimes Y = h_o^*|X\otimes Y = x_o^*\otimes y_o^*|X\otimes Y$. This implies that $x_1^*\otimes y_1^* = x_o^*\otimes y_o^*$ even on H. We conclude that $\Phi = \{x_o^*\otimes y_o^*\}$ is an extreme point of B_{H^*}.

5. GEOMETRY OF THE DUAL UNIT BALL OF $K_{w^*}(X^*,Y)$ AND DIFFERENTIABILITY OF THE NORM

After discovering that the extreme points of the dual unit ball of $K_{w^*}(X^*,Y)$ are made up exactly by the set of functionals $x^*\otimes y^*$ with x^* and y^* extreme points in the dual unit balls of X and Y, respectively, it was only natural to ask whether analogous results hold for the various special classes of extreme points. C. Stegall and I thus investigated the form of the [w*-]exposed and [w*-]strongly exposed points of the dual unit ball of $K_{w^*}(X^*,Y)$. And, finally, Joe Diestel posed the problem of determining the denting points.

We shall present in this section the respective results and discuss various applications.

First, we briefly recall the definitions of these special classes of extreme points. For a detailed discussion and equivalent definitions, the reader is referred to Joe Diestel's Lecture Notes [9] and, concerning denting points, to Phelps [44].

Definition: Let Z be a Banach space, and $z_o \in Z$ with $\|z_o\| = 1$.

(a)(i) z_o is an *exposed point* of B_Z if there exists $z_o^* \in Z^*$, $\|z_o^*\|=1$, such that $z_o^*(z_o)=1$ and $z_o^*(z) < 1$ for all $z \in B_Z \setminus \{z_o\}$.
The set of exposed points of B_Z will be denoted by $expB_Z$.

(ii) If $Z = X^*$ and the exposing functional z_o^* happens to be an element of X : $z_o^*=x_o \in X$, then z_o is said to be a *w*-exposed point* of B_{X^*}. The set of all such points will be denoted by $w^*\text{-}expB_{X^*}$.

(b)(i) z_o is a *strongly exposed point* of B_Z if there exists $z_o^* \in Z^*$, $\|z_o^*\|=1$, such that $z_o^*(z_o)=1$ and, whenever $(z_n)_n$ in B_Z is such that $z_o^*(z_n) \longrightarrow 1$, then $\|z_n - z_o\| \longrightarrow 0$.
The set of strongly exposed points of B_Z will be denoted by $sexpB_Z$.

(ii) If $Z = X^*$ and the strongly exposing functional z_o^* happens to be an element of X : $z_o^* = x_o \in X$, then z_o is said to be a *w*-strongly exposed point* of B_{X^*}. The set of all such points will be denoted by $w^*\text{-}sexpB_{X^*}$.

(c)(i) z_o is a *denting point* of B_Z if, given any $\varepsilon > 0$, there exist $\delta > 0$ and $z_\varepsilon^* \in Z^*$, $\|z_\varepsilon^*\|=1$, such that $z_\varepsilon^*(z_o) > 1-\delta$ and, whenever $\|z\| \le 1$, and $z_\varepsilon^*(z) > 1-\delta$, then $\|z-z_o\| < \varepsilon$ (i.e. z_o is contained in slices of arbitrarily small diameter).
The set of all denting points of B_Z will be denoted by $dentB_Z$.

(ii) If $Z = X^*$ and the slices of (c)(i) can always be chosen to be generated by elements $z_\varepsilon^* = x_\varepsilon$ in X, then z_o is said to be a *w*-denting point* of B_{X^*}. The set of all such points will be denoted by $w^*\text{-}dentB_{X^*}$.

5.1 Theorem [53]: *Let H be a closed linear subspace of* $K_{w^*}(X^*,Y)$, *containing* $X \tilde{\otimes}_\varepsilon Y$. *Then we have:*

(a) $[w^*\text{-}]expB_{H^*} = [w^*\text{-}]expB_{X^*} \otimes [w^*\text{-}]expB_{Y^*}$.

(b) $[w^*\text{-}]sexpB_{H^*} = [w^*\text{-}]sexpB_{X^*} \otimes [w^*\text{-}]sexpB_{Y^*}$.

(c) $[w^*\text{-}]dentB_{H^*} = [w^*\text{-}]dentB_{X^*} \otimes [w^*\text{-}]dentB_{Y^*}$.

(For the w*-part of proposition (a), compare J.A. Johnson [32].)

Thus, in the vein of our general theme set at the beginning of this paper, in all cases, the geometry of the dual unit balls of X and Y completely determines the geometry of the dual unit ball of the operator space $K_{w^*}(X^*,Y)$.

Although the results of Theorem 5.1 qualitatively are all of the same nature, the respective methods of proof they required turned out to be quite different from each other and rather involved, so that it would go beyond the scope of this survey to present even the ideas of proof. Instead, the reader is referred to our paper [53], where more general results are derived. Some of these will be quoted below.

We shall now discuss some consequences.

5.2 SOME EXTENSIONS AND CONSEQUENCES

We first use Theorem 5.1 (b) to derive Feder / Saphar's extension of Schatten's result that, for H infinite-dimensional Hilbert, K(H) is not isometric to a dual space.

5.2.1 Theorem (Feder / Saphar [14,Thm.2]): *Assume that X and Y are reflexive, and let H be any closed linear subspace of K(X,Y), containing the finite-rank operators.*

If H is isometric to a dual space, then H is reflexive.

Proof: Assume that $H = Z^*$. According to our assumptions, $Z^{**} = H^*$ has RNP (Theorem 3.2.1 above). Hence, we have (taking into account that X and Y are reflexive): $B_{Z^{**}} = \text{norm-cl co}(\text{sexp}B_{Z^{**}})$. We now use Theorem 5.1 (b) (and the fact that X and Y are reflexive) to conclude that

$$B_{Z^{**}} = \text{norm-cl co}(\text{sexp}B_{Z^{**}}) = \text{norm-cl co}(\text{sexp}B_{H^*}) =$$
$$= \text{norm-cl co}(\text{sexp}B_{X^{**}} \otimes \text{sexp}B_{Y^*}) =$$
$$= \text{norm-cl co}(w^*\text{-sexp}B_{X^{**}} \otimes w^*\text{-sexp}B_{Y^*}) =$$
$$= \text{norm-cl co}(w^*\text{-sexp}B_{H^*}) = \text{norm-cl co}(w^*\text{-sexp}B_{Z^{**}}) =$$
$$= \text{norm-cl co}(\text{sexp}B_Z) = B_Z .$$

(We used the fact that, for a general Banach space W, $w^*\text{-sexp}B_{W^{**}} = \text{sexp}B_W$.) This shows that Z, and thus H, is reflexive.

To deduce further consequences, we first recall a classical result by Smul'yan connecting the exposed point structure of the dual unit ball with differentiability properties of the norm.

Theorem (Smul'yan [64,65]): *Let z_o be an element of norm one of a Banach space Z. Then the norm of Z is Gateaux- (resp. Frèchet-) differentiable at z_o with differential z_o^* if and only if z_o w*-exposes (resp. w*-strongly exposes) B_{Z^*} at z_o^* .*

In our paper [53], instead of Theorem 5.1 (b), we prove a more general result which allows us to extend the assertion of this proposition to any linear subspace H even of $L_{W^*}(X^*,Y)$. In particular, we can deduce the following result on the geometry of the dual unit balls of K(X,Y) and L(X,Y).

5.2.2 Corollary [53]: *Let $X^* \otimes Y \subset H \subset L(X,Y)$.*
Then $w^*\text{-sexp}B_{H^*} = \text{sexp}B_X \otimes w^*\text{-sexp}B_{Y^*}$. In particular, we have:
$w^*\text{-sexp}B_{K(X,Y)^*} = w^*\text{-sexp}B_{W(X,Y)^*} = w^*\text{-sexp}B_{L(X,Y)^*} = \text{sexp}B_X \otimes w^*\text{-sexp}B_{Y^*}.$

Together with a result on G-differentiability that, again, is stronger than Theorem 5.1 (a), the extension of Theorem 5.1 (b) allows us to derive the following consequences.

5.2.3 Corollary [53]: *Let* $x_o \in X$, $y_o \in Y$, *with* $\|x_o\|=1=\|y_o\|$.

(a) Let $X \otimes Y \subset H \subset B(X^*,Y^*) = L(X^*,Y^{**})$.

Then $\| \ \|_H$ *is F-differentiable at* $x_o \otimes y_o$ *if and only if* $\| \ \|_X$ *and* $\| \ \|_Y$ *are F-differentiable at* x_o *and* y_o, *respectively.*

(b) Let $X \otimes Y \subset H \subset C_{odd}(B_{X^*} \times B_{Y^*})$ ($f \in C_{odd}(B_{X^*} \times B_{Y^*})$: $f(-x^*,y^*) = f(x^*,-y^*) = -f(x^*,y^*)$).

Then $\| \ \|_H$ *is G-differentiable at* $x_o \otimes y_o$ *if and only if* $\| \ \|_X$ *and* $\| \ \|_Y$ *are G-differentiable at* x_o *and* y_o, *respectively.*

(For proposition (b), compare J.A. Johnson [32].)

The extensions of Theorem 5.1 (a) and (b) can also be used to derive the results on F- and G-differentiability of the norms in K(X,Y) and L(X,Y) announced by S. Heinrich in [23]. We note here one particular case.

5.2.4 Corollary (Heinrich): *Let* $k_o \in K(X,Y)$ *with* $\|k_o\| = 1$. *Then the following propositions are equivalent:*

(a) The norm of $K(X,Y)$ *is F-differentiable at* k_o.

(b) The norm of $L(X,Y)$ *is F-differentiable at* k_o.

(c) The norm of $L(X^{**},Y^{**})$ *is F-differentiable at* k_o^{**}.

(Actually, Heinrich [23,Cor.4.2] proved this up to the space L(X,Y), i.e. the equivalence of (a) and (b).)

Some of the above results are significantly different from those of the preceding sections in that they transfer properties of X^* and Y^* not only to the dual of $K_{w^*}(X^*,Y)$ but to the duals of much larger spaces of merely continuous linear operators. Further results in this direction can be found in the joint paper [53] with C. Stegall.

6. PROBLEMS

We close this paper with some of the problems that naturally arise from our discussion of the operator spaces K(X,Y) and $K_{w^*}(X^*,Y)$ in the previous sections.

6.1 When is it true that K(X,Y) does not contain (an isomorph of) c_o ?

6.2 Under which conditions does K(X,Y) have RNP ?

6.3 Is there an isomorphic version of Feder / Saphar's result 5.2.1 above, i.e. if, for reflexive Banach spaces X and Y, the space K(X,Y) is isomorphic to a dual space, is K(X,Y) then necessarily reflexive ?

6.4 Instead of $K_{w*}(X^*,Y)$ - and thus $X\tilde{\otimes}_\varepsilon Y$ and $K(X,Y)$ - consider the completed projective tensor product $X\tilde{\otimes}_\pi Y$ of Banach spaces X and Y: to what extent can the geometric and topological properties of $X\tilde{\otimes}_\pi Y$ and its dual $L(X,Y^*)$ be recovered from the corresponding properties of the factor spaces X and Y and their duals ?

6.5 To what extent can geometric and topological properties of particular operator spaces other than K(X,Y) - like p-absolutely summing, p-integral, or p-nuclear operators - be recovered from those of the factor spaces X and Y and their duals ?

COMMENTS AND RELATED PROBLEMS

6.1 and 6.2: The general feeling is that - like for reflexivity itself - for positive results on these weakenings of reflexivity for K(X,Y), the coincidence of L(X,Y) with K(X,Y) will play an important role.

For the Radon-Nikodym property, this is being discussed in Diestel / Morrison [8]. They showed:

Theorem (Diestel / Morrison [8]): *Suppose that X^* is separable or reflexive, and that Y has RNP. Then, whenever L(X,Y) = K(X,Y), K(X,Y) has RNP.*

For extensions of this result, see Andrews [1]. Compare also the discussion of the RNP for operator spaces in Diestel / Uhl [10,Ch.VIII,p.258].

For the problem of containing c_0, results by A.E. Tong [68] and N.J. Kalton [35] indicate a strong connection with the coincidence of bounded and compact linear operators.

Theorem (Kalton [35]): *Suppose that X has an unconditional finite-dimensional expansion of the identity. Then, if Y is any infinite-dimensional Banach space, the following are equivalent:*

(a) $K(X,Y) \not\supset c_0$. *(b) L(X,Y) = K(X,Y) .*

(c) K(X,Y) is complemented in L(X,Y) .

Thus, problems 6.1 and 6.2 bring our attention back to the "old" problem of characterizing spaces X and Y for which there exist non-compact bounded linear operators from X into Y, and to the problem, whether K(X,Y) ever is complemented in L(X,Y) in a "non-trivial" way : is it true that either L(X,Y) = K(X,Y), or K(X,Y) is uncomplemented in L(X,Y) ?

For a thorough recent discussion of these latter problems, the reader is referred to J. Johnson [33].

6.3: Problem 6.3 has earlier been raised by J.R. Retherford [48,p.1006]. It leads us to the general problem of when K(X,Y) is (isometric or isomorphic to) a dual. In particular: is K(X) ever a dual ?

J. Hennefeld [27,Cor.2.3] showed: *If X has a complemented subspace with an unconditional basis, then K(X) is not isomorphic to a dual.*

And J. Johnson [33,Prop.1] proved the following result on the non-conjugacy of K(X,Y) :

If Y has the bounded approximation property, and K(X,Y) is not complemented in L(X,Y), then K(X,Y) is not isomorphic to a complemented subspace of a dual space.

Note that, again, an assumption on the relative position of K(X,Y) in L(X,Y) interferes !

For further information on the problem of non-conjugacy of K(X,Y) or L(X,Y), the reader may consult Hennefeld [27] and J. Johnson [33].

6.4: This program particularly includes the problem of characterizing conditions on X and Y such that $X \tilde{\otimes}_\pi Y$ has RNP, does not contain an isomorph of l_1 or c_o, or is weakly sequentially complete.

Note that this program is quite ambitious, for it includes the case of the space $L^1(\mu,X) = L^1(\mu) \tilde{\otimes}_\pi X$, which, just for the special case of weak sequential completeness and of characterizing weak compactness, turned out to be rather difficult. (As far as I know, M. Talagrand finally showed that $L^1(\mu,X)$ is weakly sequentially complete whenever X is.)

A further particular problem is to find out under which conditions the space N(X,Y) of nuclear operators has RNP; compare the remarks following Corollary 3.2.2 in section 3 above.

In [24], Heinrich showed that $\text{sexp}B_{X \tilde{\otimes}_\pi Y} = \text{sexp}B_X \otimes \text{sexp}B_Y$. We derive this result in [53] as a further special case of our extended version of Theorem 5.1 in section 5 above.

6.5: For this program, we only give a partial list of references related to this problem.

Reflexivity of the space of p-absolutely summing operators ($1 \leq p < \infty$) is being discussed in Gordon/Lewis/Retherford [17], and in Saphar [56].

D.R. Lewis [38] discussed reflexivity, the RNP, and weak sequential completeness for various classes of α-tensor products, and specializations to corresponding classes of absolutely summing and integral operators.

Heinrich [25] obtained conditions under which the spaces of p-integral and of p-absolutely summing operators (a) have RNP, and (b) do not contain (an isomorph of) c_o.

Non-containment of l_1 and of c_o, and the RNP and weak sequential completeness for the spaces of p-nuclear and quasi-p-nuclear operators are being discussed in Makarov/Samarskii [41].

Andrews [1], too, has results on the RNP for the spaces of p-absolutely summing and of p-nuclear operators.

Finally, Heinrich [26] also obtained results on the weak sequential completeness of the spaces $X \tilde{\otimes}_{g_p} Y$ and $X \tilde{\otimes}_{\varepsilon_p} Y$ [55] and of the spaces of p-absolutely summing and of p-nuclear operators.

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ON CERTAIN LOCALLY CONVEX TOPOLOGIES ON BANACH SPACES

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1. INTRODUCTION

Besides its norm topology, an infinite dimensional Banach space X carries infinitely many different locally convex topologies which are compatible with $\langle X,X'\rangle$, the duality given by X and its continuous dual X'. The most familiar example is certainly the weak topology $\sigma(X,X')$; another one is obtained by the topology of uniform convergence on the compact subsets of X'. The latter is also known to be the finest Schwartz topology on X which is compatible with $\langle X,X'\rangle$; similarly, one may consider the finest nuclear topology on X which is compatible with $\langle X,X'\rangle$, etc.

All these topologies on X may be defined by means of seminorms such that the quotient map from X to (the completion of) X modulo the kernel of such a seminorm belongs to a prescribed ideal of operators. The present note is devoted to the discussion of certain aspects of this general setting as well as to the consideration of additional examples. We will solve the problem of completeness of the locally convex topologies under consideration for a fairly large class of operator ideals, and we will describe the effect of replacing the basic ideal by its uniform closure in terms of a known procedure to generate new locally convex topologies from given ones. The question of consistency for our topologies, when passing to subspaces, will be shown to be linked with a classical Hahn-Banach situation. In the interesting case where the basic ideal consists of all $\mathcal{L}_2$-factorable operators, this link will enable us to give another description of the so-called Hilbert-Schmidt spaces [9]. We shall make use of the occasion to include some sequential characterizations of these spaces, in terms of the topologies in question, and finally we shall discuss some elementary aspects of the (unsolved) problem of whether or not there exists any reflexive Hilbert-Schmidt space of infinite dimension.

2. NOTATION

Our terminology and notation concerning Banach spaces will be more or less standard. In particular, subspaces of Banach spaces are always understood to be linear and closed submanifolds, and by an operator between two Banach spaces we mean a continuous linear map. We reserve X,Y,Z,... to denote Banach spaces, and R,S,T,... to denote operators acting between these spaces. If X is a Banach space, B_X will be its closed unit ball, X' its dual, X" its bidual, $e_X: X \longrightarrow X''$ the evaluation map, X_∞ the space $\ell_\infty(B_{X'})$, and J_X the canonical isometric embedding $X \longrightarrow X_\infty$. Let us agree that all locally convex topologies occuring in this note are understood to be Hausdorff. Any unexplained terminology concerning locally convex spaces can be found in [8]. If E is a locally convex space and U is an absolutely convex 0-neighbourhood in E, whose gauge functional is q, then we write $E_{(U)}$ for the Banach space obtained by completing the space $E/q^{-1}(0)$ with respect to the norm derived from q. By ϕ_U we denote the canonical continuous linear map $E \longrightarrow E_{(U)}$. If $V \subset U$ is another absolutely convex 0-neighbourhood in E, then ϕ_U factors through ϕ_V, i.e. there is a unique operator $\phi_{UV}: E_{(V)} \longrightarrow E_{(U)}$ such that $\phi_U = \phi_{UV} \cdot \phi_V$.

Facts and terminology from the theory of operator ideals will be used freely in the sequel. Our main reference is of course [16]; see also [8].

3. LOCALLY CONVEX TOPOLOGIES ON BANACH SPACES AND OPERATOR IDEALS

In the sequel we will briefly review a connection between ideals of operators and certain locally convex topologies on Banach spaces. The details were meticuously worked out in [21].

Suppose we are given any method to produce a locally convex topology $\mathcal{T}$ on every Banach space X, which is coarser than the norm topology and such that the passage from X to $X_\mathcal{T} := [X,\mathcal{T}]$ behaves "functorial" in the sense that $\mathcal{L}(X,Y) \subset \mathcal{L}(X_\mathcal{T}, Y_\mathcal{T})$ holds for every pair (X,Y) of Banach spaces. Then X and $X_\mathcal{T}$ obviously have the same dual, so that even $\mathcal{L}(X,Y) = \mathcal{L}(X_\mathcal{T}, Y_\mathcal{T})$ is true.

If we now put $\mathcal{A}(X,Y) := \mathcal{L}(X_{\mathcal{T}},Y)$, for all X, Y, then we get an ideal $\mathcal{A}$ of operators. Moreover, on every Banach space X, the topology $\mathcal{T}$ can be recovered from $\mathcal{A}$. In fact, the seminorms

$$p_T : X \longrightarrow \mathbb{R} : x \mapsto \| Tx \| ,$$

where T varies over $\mathcal{A}(X,Y)$ and Y runs through all, or sufficiently many, Banach spaces, form a directed generating family of seminorms for the topology $\mathcal{T}$. We note that the ideal $\mathcal{A}$ obtained in this manner is injective.

Conversely, if we are given any ideal of operators, $\mathcal{A}$, then the seminorms p_T, $T \in \mathcal{A}(X,\cdot)$, as defined above, generate a locally convex topology on X which is compatible with $\langle X,X' \rangle$ and which will be denoted by

$$\mathcal{T}_{\mathcal{A}}$$

in the sequel. Let us also agree to write

$$X_{\mathcal{A}}$$

in place of $[X,\mathcal{T}_{\mathcal{A}}]$. Since $\mathcal{L}(X,Y)=\mathcal{L}(X_{\mathcal{A}},Y_{\mathcal{A}})$ is then trivially true for all Banach spaces X and Y, the construction described above yields an ideal which is easily seen to be nothing else but the injective hull, $\mathcal{A}^{inj}$, of $\mathcal{A}$, i.e. the smallest injective ideal which contains $\mathcal{A}$.

In this way, a one-to-one correspondence is obtained between the injective operator ideals on one side and the "functorial" locally convex topologies on the other side.

4. SOME ADDITIONAL PROPERTIES OF THE TOPOLOGIES $\mathcal{T}_{\mathcal{A}}$

For a large class of operator ideals $\mathcal{A}$, the completion of $X_{\mathcal{A}}$ for a given Banach space X admits a surprisingly simple description. Let $\mathfrak{W}$ be the ideal of all weakly compact operators, and let $\mathcal{N}_\Lambda$ be the ideal of all operators $S:X \longrightarrow Y$ such that $S = A \cdot B \cdot C$ with $A \in \mathcal{L}(\ell_2,Y)$, $C \in \mathcal{L}(X,\ell_2)$, and $B \in \mathcal{L}(\ell_2,\ell_2)$ a Λ-nuclear operator, where Λ is, for example, a nuclear power series space of infinite type. Λ-nuclearity of B means that the sequence of singular numbers (approximation numbers) of B belongs to Λ .

The following extends a construction in [6]:

<u>1. Proposition:</u> Let $\mathcal{A}$ be any ideal of operators such that $\mathcal{N}_\Lambda \subset \mathcal{A} \subset \mathfrak{W}$. Then, for every Banach space X, the completion of $X_{\mathcal{A}}$ is linearly isomorphic to X".

Consequently, $X_{\mathfrak{A}}$ is complete if and only if X is reflexive.

Proof. We may suppose X to be infinite dimensional. Extending a result in [22] it was shown in [4] that X' can be represented as the locally convex inductive limit of the directed family $(H_\alpha)_{\alpha\in A}$ of Hilbert spaces embedding continuously into X' such that H_α is a linear subspace of H_β with Λ- nuclear linking mapping, for $\alpha \leqq \beta$:

$$X' = \operatorname{ind}_{\alpha\in A} H_\alpha .$$

A fortiori we may write

$$X' = \operatorname{ind}_{\gamma\in\Gamma} R_\gamma ,$$

$(R_\gamma)_{\gamma\in\Gamma}$ being the family of all reflexive Banach spaces which embed continuously into X'.

Let now $z\in X''$ be given. Then the restriction of z to B_{R_γ} is weakly continuous, and hence weak*-continuous since B_{R_γ} is weakly compact, $\forall\gamma\in\Gamma$. By Grothendieck's completeness theorem, z belongs to the completion $(X_{\mathfrak{W}})^\sim$ of $X_{\mathfrak{W}}$.

Clearly, $(X_{\mathfrak{W}})^\sim$ is a linear subspace of $(X_{\mathfrak{N}_\Lambda})^\sim$. Thus it remains to show that every $z\in(X_{\mathfrak{N}_\Lambda})^\sim$ can be regarded as an element of X'' . Again by Grothendieck's completeness theorem, z may be regarded as a linear form on X' whose restriction to B_{H_α} is weak*-continuous, $\forall\alpha\in A$. In particular, $z(B_{H_\alpha})$ is bounded, hence $z|_{H_\alpha}$ is continuous, $\forall\alpha\in A$. Thus $z\in(\operatorname{ind}_{\alpha\in A} H_\alpha)' = X''$. ●

We do not know if there is any constructive way to produce minimal operator ideals $\mathfrak{A}$ such that $(X_{\mathfrak{A}})^\sim = X''$ holds for every Banach space X.

Our next proposition establishes a connection between the topologies $\mathfrak{T}_{\mathfrak{A}}$ and $\mathfrak{T}_{\bar{\mathfrak{A}}}$, $\mathfrak{A}$ a given ideal. As usual, $\bar{\mathfrak{A}}$ denotes the ideal whose component for a given pair (X,Y) of Banach spaces is obtained by taking the closure of $\mathfrak{A}(X,Y)$ in $\mathfrak{L}(X,Y)$, with respect to the usual operator norm topology.

2. Proposition: On every Banach space X, $\mathfrak{T}_{\bar{\mathfrak{A}}}$ is the finest locally convex topology which coincides with $\mathfrak{T}_{\mathfrak{A}}$ on B_X.

Proof. From $\mathfrak{A}\subset\bar{\mathfrak{A}}$ we get $\mathfrak{T}_{\mathfrak{A}} \leq \mathfrak{T}_{\bar{\mathfrak{A}}}$.

Now let p be a continuous seminorm on X for the topology $\mathfrak{T}_{\bar{\mathfrak{A}}}$. We may assume that it is of the form $p(x)=\|Tx\|$, for some $T\in\bar{\mathfrak{A}}(X,Z)$. Let $S\in\mathfrak{A}(X,Z)$ be such that $\|Tx\|\leq\|Sx\|+\frac{1}{2}\|x\|$, $\forall x\in X$. If we put $V_S := \{x\in X \mid \|Sx\|\leq 1\}$, etc., then $B_X\cap\frac{1}{2}V_S\subset V_T$ follows, showing that $B_X\cap V_T$ is a zero-neighbourhood in $[B_X,\mathcal{T}_{\mathfrak{A}}]$. Consequently, $\mathcal{T}_{\mathfrak{A}}$ and $\mathcal{T}_{\bar{\mathfrak{A}}}$ coincide on B_X.

Next let $\mathcal{T}_o$ be the finest locally convex topology on X which coincides with $\mathcal{T}_{\mathfrak{A}}$ on B_X. A 0-basis for $\mathcal{T}_o$ is given by all sets of the form $U=\mathrm{acx}\bigcup_{n=1}^{\infty} n\cdot B_X\cap U_n$, the U_n running through a 0-basis for $\mathcal{T}_{\mathfrak{A}}$, cf. [8], 12.3. We may assume $U_n=V_{S_n}$, with suitable $S_n\in\mathfrak{A}(X,Z_n)$, so that $\|\phi_U x\|\leq\|S_n x\|+\frac{1}{n}\cdot\|x\|$ follows, $\forall x\in X$, $\forall n\in\mathbb{N}$. According to [8],20.7, this implies $\phi_U\in\bar{\mathfrak{A}}^{inj}$, and hence U is a 0-neighbourhood for $\mathcal{T}_{\bar{\mathfrak{A}}}$. ●

In the language of [8], 12.4, $\mathcal{T}_{\bar{\mathfrak{A}}}$ is the gDF-topology associated with $\mathcal{T}_{\mathfrak{A}}$.

The next proposition is included merely for the sake of completeness.

3. Proposition: Let $\mathfrak{A}$ be a surjective ideal, X a Banach space, and Y a subspace of X. Then $(X/Y)_{\mathfrak{A}}$ carries the quotient topology of $X_{\mathfrak{A}}$.

Proof. All we need to show is that the quotient map $Q:X_{\mathfrak{A}}\longrightarrow(X/Y)_{\mathfrak{A}}$ is open. For this, let U be a 0-neighbourhood in $X_{\mathfrak{A}}$. We may assume $U=V_T$ for some $T\in\mathfrak{A}(X,Z)$. Let Z_a be the closure of the space $T(Y)$. Let $Z_b := Z/Z_a$ and $Q_b:Z\longrightarrow Z_b$ be the corresponding quotient map. By surjectivity of $\mathfrak{A}$, we may write $Q_b\circ T = S\circ Q$ for some $S\in\mathfrak{A}(X/Y, Z_b)$. Consider the 0-neighbourhood V_S in $(X/Y)_{\mathfrak{A}}$. The proof is complete if we can show $\frac{1}{2}V_S\subset Q(U)$. But if $Qx\in\frac{1}{2}V_S$, then $\|SQx\|=\|Q_bTx\|\leq\frac{1}{2}$, so that $\|Tx+z\|<1$ for some $z\in Z_a$, hence $\|Tx+Ty\|<1$, or $x+y\in U$, for some $y\in Y$. Thus $Qx\in Q(U)$. ●

A corresponding result for subspaces cannot be obtained by simply considering injective ideals: We know that $\mathcal{T}_{\mathfrak{A}}=\mathcal{T}_{\mathfrak{B}}$ holds if $\mathfrak{A}$ and $\mathfrak{B}$ have the same injective hull. Let us look a little closer to this situation.

Let X and Y be Banach spaces such that X is a subspace of Y. Suppose $X_{\mathfrak{A}}$ is a subspace of $Y_{\mathfrak{A}}$. It is then easy to see that, given any $S\in\mathfrak{A}(X,Z)$, there exists a Banach space Z_o and an operator $T\in\mathfrak{A}(Y,Z_o)$ such that $\|Sx\|\leq\|Tx\|$, $\forall x\in X$. A simple argument (cf. [21] and [16]) yields the existence of an operator $R\in\mathcal{L}(Z_o,Z_\infty)$ such that $R\cdot T\cdot I=J_Z\cdot S$, I being the embedding $X \longrightarrow Y$. Consequently $R \circ T \in \mathfrak{A}$ extends $J_Z \circ S \in \mathfrak{A}$.

The converse is also true: If every $S \in \mathfrak{A}(X,Z)$ can be extended as indicated, then $X_{\mathfrak{A}}$ is obviously a subspace of $Y_{\mathfrak{A}}$. We have thus proved that our problem is equivalent to a Hahn-Banach extension property for $\mathfrak{A}$-operators:

4. Proposition: Let $\mathfrak{A}$ be an operator ideal, and let X and Y be Banach spaces such that X is isomorphic to a subspace of Y, via an isomorphic embedding I. Then $X_{\mathfrak{A}}$ is isomorphic to a subspace of $Y_{\mathfrak{A}}$ if and only if, for every Banach space Z and every operator $S \in \mathfrak{A}(X,Z)$, there exists an operator $T \in \mathfrak{A}(Y,Z_\infty)$ such that $T \circ I = J_Z \circ S$.

5. Corollary: Let $\mathfrak{A}$ be an ideal of operators. $X_{\mathfrak{A}}$ is a subspace of $Y_{\mathfrak{A}}$ whenever X is a subspace of Y if and only if $\mathfrak{A}^{inj}(X,Z)$ and $(\mathfrak{A} \circ \Gamma_\infty)^{inj}(X,Z)$ coincide for all Banach spaces Z.

Here Γ_p, $1 \leq p \leq \infty$, denotes the ideal of all operators $S: Z_a \longrightarrow Z_b$ such that $e_{Z_b} \circ S: Z_a \longrightarrow Z_b''$ factors through an $\mathcal{L}_p(\mu)$ space. Note that in case p=2 the passage to Z_b'' can be omitted.

The hypothesis in 5 is satisfied if $\mathfrak{A}$ contains Γ_∞ as a right factor, i.e. if $\mathfrak{A} = \mathfrak{B} \circ \Gamma_\infty$ holds with some ideal $\mathfrak{B}$.

5. SOME APPLICATIONS

If in 4.5, we take the ideal $\mathfrak{K}$ of all compact operators for $\mathfrak{A}$, then the assumptions made there are satisfied, because $\mathfrak{K}$ equals $[\mathfrak{K} \circ \Gamma_\infty]^{inj}$. On a Banach space X, $\mathcal{T}_{\mathfrak{K}}$ is nothing else but the finest Schwartz topology which is compatible with $\langle X, X' \rangle$.

Let $\mathcal{N}_{1,2,2}$ be the smallest ideal whose components on Hilbert spaces are formed by the corresponding trace class operators. On a Banach space X, the topology $\mathcal{T}_{\mathcal{N}_{1,2,2}}$ is the finest nuclear topology which is compatible with $\langle X, X' \rangle$. It is easy to see that $\mathcal{T}_{\mathcal{N}_{1,2,2}} = \mathcal{T}_{\mathcal{N}_{1,2,2} \circ \Gamma_\infty}$ and that 4.5 applies to $\mathcal{N}_{1,2,2} \circ \Gamma_\infty$.

The ideal $\mathcal{N}_1$ of all nuclear operators also satisfies the hypothesis of 4.5. We shall consider the spaces $X_{\mathcal{N}_1}$ a little later (cf. 7.3.).

The reader may easily extend the list of ideals verifying the hypothesis of 4.5. We shall now change our approach slightly and discuss restrictions to be imposed on the spaces in question, in order to make 4.4 work.

To clarify what we have in mind, consider, for example, the ideal $\mathfrak{W}$ of all weakly compact operators. Given a Banach space X, the space $X_{\mathfrak{W}}$ is a subspace of $Y_{\mathfrak{W}}$ whenever X is a subspace of Y if and only if the identiy I_X of X belongs to the quotient ideal $\mathfrak{W}^{-1} \circ [\mathfrak{W} \circ \Gamma_\infty]^{inj}$. This is equivalent to saying that for every Banach space Z_o and every $S \in \mathfrak{W}(X, Z_o)$ there exists a Banach space Z containing Z_o as a subspace and a Z-valued measure μ on the Borel sets of $[B_{X'}, \sigma(X', X)]$ such that S is "Riesz representable" by μ, in the sense that $Sx = \int_{B_{X'}} \langle a, x \rangle d\mu(a)$ holds in Z, $\forall x \in X$. It is known that, besides $\mathcal{L}_\infty$-spaces, certain spaces of analytic functions, such as the disc algebra, also have this property, cf. [10].

We next consider the ideal Γ_2 of all operators which factor through a Hilbert space. Let us denote the ideal of all absolutely p-summing operators by $\mathfrak{P}_p$, $1 \leq p < \infty$.

<u>1. Proposition:</u> X has the property that X_{Γ_2} is a subspace of Y_{Γ_2} whenever X is a subspace of Y if and only if $\mathfrak{L}(X, H) = \mathfrak{P}_2(X, H)$ holds for every Hilbert space H.

Of course, it suffices to take the space ℓ_2 for H. Note that X_{Γ_2} carries the finest locally convex topology which is hilbertisable and compatible with $\langle X, X' \rangle$.

The proposition is a simple corollary of 4.4 and the well-known fact that $\mathfrak{P}_2 = \Gamma_2 \circ \Gamma_\infty$ holds. If X satisfies $\mathfrak{L}(X, \ell_2) = \mathfrak{P}_2(X, \ell_2)$, then every operator on ℓ_2 which factors through X is a Hilbert-Schmidt operator. By appealing to Dvoretzky's theorem [3], it was shown in [1] that every compact operator on ℓ_2 factors through a subspace of an arbitrarily given infinite dimensional Banach space X; hence every Hilbert-Schmidt operator on ℓ_2 factors through X itself. Consequently, the infinite dimensional Banach spaces X, such that $\mathfrak{L}(X, \ell_2) = \mathfrak{P}_2(X, \ell_2)$, are characterized by the property that the operators on ℓ_2 which factor through X are precisely the Hilbert-Schmidt operators. It is for this reason that these spaces were called Hilbert-Schmidt spaces in [9].

Here are some elementary characterizations of Hilbert-Schmidt spaces; cf. [9]:

2. Proposition: For every Banach space X, the following are equivalent:

(a) X is a Hilbert-Schmidt space.

(b) If (x_n) and (a_n) are weak ℓ_2-sequences in X and X' respectively, then $(\langle a_n, x_n\rangle) \in \ell_2$.

(c) X' is a Hilbert-Schmidt space.

By a result of Grothendieck (cf. [5] and [14]), $\mathcal{L}_1$-spaces and $\mathcal{L}_\infty$-spaces are Hilbert-Schmidt spaces. But there are other examples, e.g. quotients of $\mathcal{L}_1$-spaces by reflexive subspaces, and the corresponding duals (cf. [13] and [17]), some spaces of analytic functions such as H_∞ and the disc algebra (see [2]), and all Banach spaces Z such that $Z \otimes_\varepsilon Z = Z \otimes_\pi Z$; see [20] for the existence of infinite dimensional Banach spaces of this type. Many of the known examples even deal with a Banach space X such that X and/or X' has cotype 2, implying that X and/or X' "verifies Grothendieck's theorem" (all operators into ℓ_2 belong to $\mathfrak{P}_1$). If X is a Hilbert-Schmidt space such that X and X' both have cotype 2, then X cannot have the approximation property, unless dim $X<\infty$; cf. [18].

6. ℓ_2-SUMMABLE SEQUENCES IN SPACES X_{Γ_2}

Further let X be a Banach space and let $\mathfrak{U}$ be a 0-basis of X_{Γ_2}. What follows does not depend on the particular choice of $\mathfrak{U}$, and therefore we may assume from the beginning that $\mathfrak{U}$ consists of all sets $V_T = T^{-1}(B_H)$, H being a sufficiently large Hilbert space and T running through $\mathcal{L}(X,H)$.

On the linear space $\ell_2(X)$ of all weak ℓ_2-sequences in X we consider the locally convex topology given by all seminorms

$$\varepsilon_T((x_n)) := \sup \{ (\sum_n |\langle a, x_n\rangle|^2)^{\frac{1}{2}} | a \in V_T^\circ \},$$

$T \in \mathcal{L}(X,H)$. The resulting locally convex space will be denoted by

$$\ell_2(X_{\Gamma_2}).$$

We write $\ell_2(X)$ if we wish to consider our space as a Banach space with respect to the norm $\varepsilon_2((x_n)) := \sup\{ (\sum_n |\langle a, x_n\rangle|^2)^{\frac{1}{2}} | a \in B_{X'} \}$.

Next we define the space

$$\ell_2\{X_{\Gamma_2}\} := \{(x_n)\varepsilon X^{\mathbb{N}} \mid (\|Tx_n\|)\varepsilon \ell_2,\ \forall T\varepsilon \mathcal{L}(X,H)\},$$

topologized by means of the seminorms

$$\alpha_T((x_n)) := (\sum_n \|Tx_n\|^2)^{\frac{1}{2}},\ T\varepsilon \mathcal{L}(X,H).$$

This space has to be distinguished from the space

$$\ell_2\{X\} := \{(x_n)\varepsilon\ X^{\mathbb{N}} \mid (\|x_n\|)\varepsilon\ \ell_2\}$$

which is a Banach space with respect to the norm

$$\alpha_2((x_n)) := (\sum_n \|x_n\|^2)^{\frac{1}{2}}.$$

Recall that $\mathcal{L}(\ell_2,X)$ and $\ell_2(X)$ are isometrically isomorphic by virtue of $S \mapsto (Se_n)$. Here e_n denotes the n-th standard unit vector in ℓ_2. It is clear from the definition that for $(x_n)=(Se_n)$ in $\ell_2(X_{\Gamma_2})$ we have $\varepsilon_T((x_n))=\|T\circ S\|$, $\forall T\varepsilon\mathcal{L}(X,H)$. Moreover (x_n) belongs to $\ell_2\{X_{\Gamma_2}\}$ iff $T\circ S:\ell_2 \longrightarrow H$ is a Hilbert-Schmidt operator, and in that case, $\alpha_T((x_n))$ is nothing else but the Hilbert-Schmidt norm of $T\circ S$, $\forall T\varepsilon\mathcal{L}(X,H)$. In other words, $S \mapsto (Se_n)$ induces a linear isomorphism from $\mathfrak{P}_2^{dual}(\ell_2,X)$ onto $\ell_2\{X_{\Gamma_2}\}$. It can be shown that this isomorphism is continuous with respect to the canonical norm, π_2^{dual}, on $\mathfrak{P}_2^{dual}(\ell_2,X)$. In fact, $\pi_2^{dual}(S)$ equals the supremum of all $\alpha_T((x_n))$, where T runs through the unit ball of $\mathcal{L}(X,H)$, $\forall S\varepsilon\mathfrak{P}_2^{dual}(\ell_2,X)$.

These remarks immediately lead us to

<u>1. Proposition:</u> A Banach space X is a Hilbert-Schmidt space if and only if $\ell_2(X_{\Gamma_2})$ and $\ell_2\{X_{\Gamma_2}\}$ coincide as linear spaces.

In fact, a little more can be proved. Let $\mathfrak{P}_{2,2,2}$ be the ideal of all operators $S:X \longrightarrow Y$ such that $(\langle Sx_n,b_n\rangle)\ \varepsilon\ \ell_2$ for all weak ℓ_2-sequences (x_n) and (b_n) in X and Y' respectively. According to [16], $\mathfrak{P}_{2,2,2}$ is the largest ideal whose components on Hilbert spaces coincide with the corresponding spaces of Hilbert-Schmidt operators. By 5.2, a Banach space X is a Hilbert-Schmidt space iff its identity I_X belongs to $\mathfrak{P}_{2,2,2}$. For arbitrary Banach spaces X and Y we have
$S\varepsilon\mathfrak{P}_{2,2,2}(X,Y) \Leftrightarrow T\circ S\varepsilon\mathfrak{P}_2(X,H)\ \forall T\varepsilon\mathcal{L}(Y,H) \Leftrightarrow (Sx_n)\varepsilon\ell_2\{X_{\Gamma_2}\}\ \forall(x_n)\varepsilon\ell_2(X_{\Gamma_2})$.

Thus $\mathfrak{P}_{2,2,2}(X,Y)$ consists exactly of those operators $X \longrightarrow Y$ which give rise to a linear mapping from $\ell_2(X_{\Gamma_2})$ into $\ell_2\{Y_{\Gamma_2}\}$. What can be said about continuity of this mapping?

2. Proposition: $S \varepsilon \mathcal{L}(X,Y)$ induces a continuous linear mapping $\ell_2(X_{\Gamma_2}) \longrightarrow \ell_2\{Y_{\Gamma_2}\}$ iff it belongs to $\mathfrak{P}_2^{dual}(X,Y)$.

Proof. Suppose firstly $S \varepsilon \mathfrak{P}_2^{dual}(X,Y)$. Then, given $T:Y \longrightarrow H$, there is a $V \varepsilon \mathcal{L}(X,H)$ and a Hilbert-Schmidt operator $W:H \longrightarrow H$ such that $W \circ V = T \circ S$. As before, H is a suitable Hilbert space. Since W induces a continuous linear map $\ell_2(H) \longrightarrow \ell_2\{H\}$, we can find a $c>0$ such that $\alpha_2((WVx_n)) \leq c \cdot \varepsilon_2((Vx_n))$, $\forall (x_n) \varepsilon \ell_2(X_{\Gamma_2})$. Equivalently, $\alpha_T((Sx_n)) \leq c \cdot \varepsilon_V((x_n))$, $\forall (x_n) \varepsilon \ell_2(X_{\Gamma_2})$. Consequently, S induces a continuous operator $\ell_2(X_{\Gamma_2}) \longrightarrow \ell_2\{Y_{\Gamma_2}\}$.

Suppose, conversely, that S induces such a continuous mapping. Thus, given $T \varepsilon \mathcal{L}(Y,H)$, we can find a Hilbert space G, an operator $V \varepsilon \mathcal{L}(X,G)$ and a $c>0$ such that $\alpha_T((Sx_n)) \leq c \cdot \varepsilon_V((x_n))$, $\forall (x_n) \varepsilon \ell_2(X_{\Gamma_2})$. This gives

$$(*) \quad \left(\sum_{i=1}^{n} \|TSx_i\|^2\right)^{\frac{1}{2}} \leq c \cdot \sup_{a \varepsilon B_G} \left(\sum_{i=1}^{n} |(a|Vx_i)|^2\right)^{\frac{1}{2}}$$

for all $x_1,..,x_n \varepsilon X$ and all $n \varepsilon \mathbb{N}$. Here we denote scalar products by $(\cdot|\cdot)$. Note that $Vx = V\tilde{x}$ implies $TSx = TS\tilde{x}$ for $x,\tilde{x} \varepsilon X$, so that $Vx \mapsto TSx$ yields a well-defined map $W:V(X) \longrightarrow H$ which is clearly linear. By (*) it is also continuous. We may assume $G = \overline{V(X)}$. The unique extension of W to an element of $\mathcal{L}(G,H)$ will also be labeled by W. Note that (*) implies

$$\left(\sum_{i=1}^{n} \|Wu_i\|^2\right)^{\frac{1}{2}} \leq c \cdot \sup_{a \varepsilon B_G} \left(\sum_{i=1}^{n} |(a|u_i)|^2\right)^{\frac{1}{2}}$$

for all $u_1,..,u_n \varepsilon G$ and all $n \varepsilon \mathbb{N}$, so that W is a Hilbert-Schmidt operator. Since T was arbitrary, we get now $S \varepsilon \mathfrak{P}_2^{dual}(X,Y)$. ●

As a counterpart of 1 we may now state:

3. Corollary: Only for Banach spaces X of finite dimension, $\ell_2(X_{\Gamma_2})$ and $\ell_2\{X_{\Gamma_2}\}$ coincide as locally convex spaces.

For the sake of completeness, we include the following elementary result:

4. Proposition: For every $S\in\mathcal{L}(X,Y)$ the following are equivalent:

(a) S maps $\ell_2\{X_{\Gamma_2}\}$ continuously into $\ell_2\{Y\}$.

(b) S maps $\ell_2\{X_{\Gamma_2}\}$ into $\ell_2\{Y\}$.

(c) S maps $\ell_2(X_{\Gamma_2})$ continuously into $\ell_2(Y)$.

(d) $S \in \Gamma_2(X,Y)$.

Proof. (a) ⇒ (b), (d) ⇒ (a), and (d) ⇒ (c) are trivial. The assumption of (b) implies $S\circ T \in \mathcal{P}_2(\cdot,Y)$ $\forall T\in\mathcal{P}_2^{\mathrm{dual}}(\cdot,X)$. Hence $S\in\Gamma_2(X,Y)$, cf. [11], i.e. (d) holds. Thus it remains to show (c) ⇒ (d). By hypothesis, we can find $T\in\mathcal{L}(X,H)$ and $c>0$ such that $\varepsilon_2((Sx_n))\leq c\cdot\varepsilon_T((x_n))$, $\forall\ (x_n)\in\ell_2(X_{\Gamma_2})$. In particular, $\|Sx\|\leq c\cdot\|Tx\|$, $\forall x\in X$. By a standard argument (used already in the proof of 2) it follows that S factors through H. ●

In particular, X is a Hilbert space ⇔ $\ell_2\{X_{\Gamma_2}\} = \ell_2\{X\}$ as linear spaces ⇔ $\ell_2(X_{\Gamma_2}) = \ell_2(X)$ as locally convex spaces. Combining this with 1 we get the classical result that $\ell_2(X) = \ell_2\{X\}$ occurs only for finite dimensional Banach spaces X.

7. REMARKS ON REFLEXIVE HILBERT-SCHMIDT SPACES

One of the unsolved problems in our context is the question if there exists a reflexive Hilbert-Schmidt space of infinite dimension. In fact, this question is open even for Banach spaces X such that X and X' both have cotype 2 and $X\otimes_\varepsilon X = X\otimes_\pi X$ holds; cf. [20].

First of all, since no Hilbert-Schmidt space can contain the ℓ_2^n's uniformly complemented, there are certainly no infinite dimensional Hilbert-Schmidt spaces of type >1; in particular, no infinite dimensional Hilbert-Schmidt space can be super-reflexive; cf. [19] and [9].

Suppose now X is a reflexive Hilbert-Schmidt space. Because of $\mathcal{L}(X,H) = \mathcal{P}_2(X,H)$, all operators from X to H are fully complete, hence compact. If $\mathcal{N}_2$ denotes the ideal of 2-nuclear operators, then, by a result of [15] and application of the closed graph theorem to the ideal norms which are involved, we even get $\mathcal{L}(X,H) = \mathcal{N}_2(X,H)$. Whereas this latter equation is also valid e.g. for the space c_o, we have, in our case, in addition $\mathcal{L}(H,X) = \mathcal{N}_2^{dual}(H,X)$ by just looking at the dual X'. It follows, for example, that X cannot have a quotient or a subspace isomorphic to ℓ_2. But we do not know if the above two equations already imply that the Hilbert-Schmidt space X is reflexive. Note however, that for every reflexive Hilbert-Schmidt space X it follows that $\mathcal{L}(X,H)$ and $\mathcal{L}(H,X)$ are reflexive, cf. [7]. Also note that we are dealing with a rather particular case of the following long-standing general problem posed by A. Pełczyński: If X and Y are Banach spaces such that $\mathcal{L}(X,Y)=\mathcal{K}(X,Y)$ and $\mathcal{L}(Y,X)=\mathcal{K}(Y,X)$, is then X or Y finite dimensional?

Let us return to reflexive Hilbert-Schmidt spaces in general.

1. Proposition: Let X be a reflexive Hilbert-Schmidt space. Then X_{Γ_2} is a complete Schwartz space, and for every infinite dimensional Banach space Y, there exists a set M such that X_{Γ_2} is isomorphic to a subspace of the product space Y^M.

Proof. Every 2-nuclear operator $X \longrightarrow H$ is the composite of another 2-nuclear operator $X \longrightarrow H$ followed by a compact operator $H \longrightarrow H$. Thus $\mathcal{L}(X,H) = \mathcal{N}_2(X,H)$ implies X_{Γ_2} is a Schwartz space. Its completeness follows from 4.1, and the embedding into Y^M for suitable M was proved in [1]. ●

Further suppose X to be a reflexive Hilbert-Schmidt space. Since every 2-nuclear operator $X \longrightarrow Y$ is the composite of another 2-nuclear operator $X \longrightarrow H$ followed by a bounded operator $H \longrightarrow Y$, we get $\mathfrak{T}_{\Gamma_2} = \mathfrak{T}_{\mathcal{N}_2}$ on X. If X is reflexive and verifies Grothendieck's theorem (i.e. $\mathcal{L}(X,H) = \mathcal{P}_1(X,H)$ holds), then we may proceed exactly in the same manner to obtain $\mathcal{L}(X,H) = \mathcal{N}_1^{inj}(X,H)$ and hence $\mathfrak{T}_{\Gamma_2} = \mathfrak{T}_{\mathcal{N}_1}$ on X. In general, $\mathfrak{T}_{\mathcal{N}_{1,2,2}} \leq \mathfrak{T}_{\mathcal{N}_1} \leq \mathfrak{T}_{\mathcal{N}_2} \leq \inf\{\mathfrak{T}_{\Gamma_2}, \mathfrak{T}_{\mathcal{K}}\}$ on every Banach space. We are interested in coincidence of $\mathfrak{T}_{\Gamma_2}$ with $\mathfrak{T}_{\mathcal{K}}$ or $\mathfrak{T}_{\mathcal{N}_{1,2,2}}$.

2. Proposition: Let X be any Banach space.

(a) If $\mathfrak{T}_{\Gamma_2}$ is finer than $\mathfrak{T}_{\mathcal{K}}$ on X, then X is a Hilbert space.

(b) If $\mathfrak{T}_{\Gamma_2} = \mathfrak{T}_{\mathcal{N}_{1,2,2}}$ on X, then $\dim X < \infty$.

Proof. (a) Our hypothesis implies that, given any compact operator $S:X \longrightarrow Y$ there is an operator $T:X \longrightarrow H$ such that $\|Sx\| \leq \|Tx\|$, $\forall x \in X$. It follows that S factors through a Hilbert space, so that $\mathcal{K}(X,Y) \subset \Gamma_2(X,Y)$ follows. Application of Pietsch-Persson duality [16], [11] yields that the 2-dominated operators with values in X are already integral, which is only possible if X is a Hilbert space.

(b) If $\mathcal{T}_{\Gamma_2}$ and $\mathcal{T}_{\mathcal{N}_{1,2,2}}$ coincide on X, then $\mathcal{L}(X,H)$ consists of nuclear operators only, implying that I_X is 2-dominated and hence $\dim X < \infty$. ●

The hypothesis in (b) implies $\mathcal{T}_{\mathcal{N}_1} = \mathcal{T}_{\mathcal{N}_{1,2,2}}$. We shall see that the conclusion of (b) also follows from this weaker assumption.

3. Proposition: Let X be a Banach space. If $X_{\mathcal{N}_1}$ is nuclear, then $\dim X < \infty$.

Proof. Our assumption implies that, given any $T \in \mathcal{N}_1(X,Y)$, there is an $S \in \mathcal{N}_1(X,Z)$ such that $V_S \subset V_T$ and $\phi_{V_T V_S}: X_{(V_S)} \longrightarrow X_{(V_T)}$ is also nuclear. Consequently, T belongs to $\mathcal{N}_{1,2,2}$. In particular, $\mathcal{N}_1(X,X) = \mathcal{N}_{1,2,2}(X,X)$ holds, showing that the eigenvalues of every nuclear endomorphism on X form an ℓ_1-sequence. By [12], X must be a Hilbert space. But it follows, for example, from [11] that for every infinite-dimensional Hilbert space H there is a Banach space Z and a nuclear operator $H \longrightarrow Z$ which does not belong to $\mathcal{N}_{1,2,2}$. Thus we get that X is finite dimensional. ●

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DECOMPOSITIONS OF POSITIVE OPERATORS AND SOME OF THEIR APPLICATIONS

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Since the work of Kakutani and Yosida on the "Operator theoretical treatment of Markoff's process" there has always been a connection between the theory of positive operators and stochastic kernels (see [55], [7]). For the most part one "translated" statements on transition probabilities into statements on positive operators and tried to apply functional analytic methods to problems of probability theory. But there are also problems in operator theory where it is advantageous to think of a positive operator T as being represented by a stochastic kernel (μ_y),

$$(*) \qquad Tf(y) = \int f \, d\mu_y ,$$

in order to be able to apply probabilistic and measure theoretical methods. Recent examples for this may be found in works of Arveson on operator algebras ([1]) and Kalton on L_p-spaces ([21], [22], [23]). In this report we emphasize the second point of view. We describe the representation (*) (Section 1) and certain decompositions connected with it (Section 2). The remaining sections are devoted to applications of this approach:

Sec. 3 L_1-subspaces of Banach lattices and the Radon-Nikodym property,

Sec. 4 Approximation by weakly compact operators in L_1,

Sec. 5 The essential spectra of L_1-operators and an application to the linear transport equation,

Sec. 6 Convolution operators.

1. THE REPRESENTATION THEOREM

Recall some of the usual examples for positive or, more generally, regular operators:

- integral operators given by a measurable kernel k

$$(1) \qquad Tf(y) = \int k(y,x)f(x)d\mu(x).$$

- Rieszhomomorphism given by a transformation σ of the underlying state space and a multiplication by a function a

$$(2) \qquad Tf(y) = a(y)f(\sigma(y)).$$

- Convolution by a measure λ on a locally compact group G

(3) $$Tf(y) = \int f(y-x)d\lambda(x)$$

- The semi group generated by the transition probabilities P_t of a Markov process

(4) $$T_t f(y) = \int dP_t(y,dx)f(x)$$

If we consider these operators as operators $T: E \to F$ in Banach function spaces E and F[1] on standard measure spaces (X,A,μ) and $(Y,\mathcal{B},\nu)$ with finite μ and ν, then they are all of the form

(5) $$Tf(y) = \int f\,d\mu_y \qquad \nu\text{-a.e.}$$

for all $f \in E$ where $(\mu_y)_{y\in Y}$ is a stochastic kernel of signed measures μ_y on (X,A) [2]. Indeed, in (1) we may use the μ-absolutely continuous kernel $d\mu_y = k(y,\cdot)d\mu$, in (2) we have the point measures $\mu_y = a(y)\cdot\delta_{\sigma(y)}$ and in (3) we put $\mu_y(A) = \lambda(A-y)$.
But besides concrete examples there is also a nice functional analytic characterization of the representation (5).

THEOREM 1. *Let* $T: E \to F$ *be dominated by a positive operator* $S: E \to F$, *i.e.* $|Tf| \leq Sf$ *for all* $0 \leq f \in E$.
Then T *has a representation* (5) *if and only if* T *is order continuous, i.e. if* $0 \leq f_n \leq f \in E$ *and* $f_n \to 0$ μ*-a.e. always imply that* $Tf_n \to 0$ ν*-a.e.*

For $E = F = L_\infty$ this is essentially Neveu's characterization of Sub-Markovian operators (see [34], § 5.4); the case $E = F = L_2$ was treated by Arveson ([1]) in the context of operator algebras on Hilbert space; for $E = F = L_1$ Kalton [21] gave an independent proof and applied the theorem to the structure theory of L_p-spaces; an even more general version is due to Sourour (see [42]).

SKETCH OF PROOF

If T is order continuous then we can extend

$$P(A\times B) = \int \chi_B \cdot T(\chi_A)d\nu, \quad A \in A,\ B \in \mathcal{B}$$

---

1) E is an order ideal in the space $L_0(X,A,\mu)$ of equivalence classes of measurable functions on (X,A,μ) containing the characteristic functions and at the same time a Banach lattice with respect to the pointwise ordering.

2) We assume that $y \in Y \to \mu_y(A)$ is a $\mathcal{B}$-measurable function for every fixed $A \in A$ but not that the μ_y are probability measures. In (5) we assume implicitly that $\int |f|d|\mu_y| < \infty$ for ν-almost all $y \in Y$ and that $\mu(A) = 0$ implies $|\mu_y|(A) = 0$ ν-a.e.

to a measure on the product space $(X\times Y, \mathcal{A}\otimes\mathcal{B})$. Disintegrating P with respect to Y gives a stochastic kernel (μ_y) of measures on $(X,\mathcal{A})$ with

$$P(U) = \iint \chi_U(x,y)d\mu_y(x)d\nu(y), \quad U\in\mathcal{A}\otimes\mathcal{B}$$

and now one can check that (μ_y) satisfies (5). The converse follows from Lebesgue's convergence theorem.

□

The main advantage of the representation (5) is certainly that it introduces certain techniques from probability theory and Banach space theory to operator theory. The decompositions discussed in the next section and the subsequent applications will be an example for this. For the moment I just mention some simple but very useful and intuitive consequences of this "pointwise" approach to positive operators.

- The modulus $|T|$ of the operator T in theorem 1, defined by

(6) $$|T|f = \sup\{|Tg| : |g|\leq f\}, \quad 0\leq f\in E$$

is represented by $(|\mu_y|)_{y\in Y}$. Similarly, if

$$Sf(y) = \int f\, d\lambda_y \qquad \nu\text{-a.e.}$$

then $S\wedge T$ and $S\vee T$ (see [41] IV. § 1) are represented by the measures $\mu_y\wedge\lambda_y$ and $\mu_y\vee\lambda_y$ resp. (see e.g. [42], [50]).

(7) - If $E=F=L_\infty$ then $\|T\| = \sup\limits_x|\mu_x|$

- (5) allows for a simple proof of the "little Riesz theorem" (cf. [41] V §8): If T is bounded as a linear operator in L_1 and L_∞ then we have for all $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $0\leq f\in L_\infty(X)$

$$\begin{aligned}(|T|f(y))^p &= (\int f(x)d|\mu_y|)^p \\ &\leq |\mu_y|(X)^{p/q}\cdot\int f(x)^p d|\mu_y| \\ &\leq \|T\|_{L_\infty}^{p/q}\cdot|T|(f^p)(y)\end{aligned}$$

and furthermore

$$\begin{aligned}\int(|T|f(y))^p d\nu(y) &\leq \|T\|_{L_\infty}^{p/q}\int|T|(f^p)(y)d\nu(y) \\ &\leq \|T\|_{L_\infty}^{p/q}\,\|T\|_{L_1}\int f^p(X)d\mu(x).\end{aligned}$$

Hence

$$\|T\|_{L_p} \leq \||T|\|_{L_p} \leq \|T\|_{L_\infty}^{1/q}\,\|T\|_{L_1}^{1/p}.$$

Somewhat surprisingly there is a converse to this interpolation result.

THEOREM 2 ([47]). *If* $T : L_p(X,\mu) \to L_p(X,\mu)$, $1 \le p < \infty$, *is dominated by a positive operator* S *in* $L_p(X,\mu)$ *then there is a positive isometry* J *of* $L_p(X,\mu)$ *such that* JTJ^{-1} *extends to a bounded operator*

$$J T J^{-1} : L_q(X,\mu) \to L_q(X,\mu)$$

for all $1 \le q \le \infty$.

There is also a somewhat weaker version of this 'change of density' result for regular operators in Banach lattices (see [47]).
These theorems may be used to reduce a general problem on positive operators to the L_q-space best suited for it (usually $q = 1$, 2 or ∞). Let us take that as an excuse to simplify the presentation by restricting ourselves to the L_1-case most of the times. One advantage here is that *every* bounded operator in L_1 has a representation (5) (because every L_1-operator is dominated by a positive operator, see [41] IV § 1).

2. DECOMPOSITIONS OF POSITIVE OPERATORS

The decomposition of a Markov operator into an absolutely continuous part and a 'singular' one is a standard method in the ergodic theory of Markov processes (e.g. see 'recurrence in the sense of Harris', [13] Chap. V). Also in Harmonic Analysis, e.g. if one wants to describe invertible convolution operators (3) as closely as possible by exponentials of the algebra M(G) it is necessary to consider singular and atomic parts of the involved measures (see [15] p. 136). In this section we will study such decompositions from a more general point of view and give further applications of this approach in the following sections. Given a stochastic kernel $(\mu_y)_{y \in Y}$ on (X,A), we use Lebesgue's decomposition to produce operators

$$(8) \qquad T^i f(y) = \int f \, d\mu_y^i , \qquad T^s f(y) = \int f \, d\mu_y^s \qquad \nu\text{-a.e.}$$

where μ_y^i is μ-absolutely continuous and μ_y^s is μ-singular and therefore $T = T^i + T^s$. By the Radon-Nikodym theorem T^i is just a classical integral operator (1) and called the integral part of T. T^s is called the singular part and we say that T has a *singular representation*, $T \in \underline{B}^s$, if $T^i = 0$. Similarly, by decomposing each μ_y into its purely atomic part μ_y^a and its diffuse part μ_y^d (i.e. $\mu_y^d(\{x\}) = 0$ for all $x \in X$) we obtain operators

$$(9) \qquad T^d f(y) = \int f \, d\mu_y^d , \qquad T^a f(y) = \int f \, d\mu_y^a$$

with $T = T^d + T^a$. T has a *diffuse* (*atomic*) *representation*, $T \in \underline{B}^d$ ($T \in \underline{B}^a$), if the atomic part T^a (diffuse part T^d) of T is zero. N. Kalton has observed in [21] that T^a can be written as

(10) $$T^a f(y) = \sum_{n=1}^{\infty} a_n(y) f(\sigma_n(y))$$

where $a_n : Y \to \mathbb{R}$ and $\sigma_n : Y \to X$ are measurable functions satisfying

$$|a_n(y)| \geq |a_{n+1}(y)| \geq \dots, \quad \sum_{n=1}^{\infty} |a_n(y)| < \infty$$

$$\sigma_n(y) \neq \sigma_m(y) \qquad \text{for } m \neq n$$

i.e. T^a is a "sum" of Rieszhomomorphisms as in (2).

To write down such decompositions is of course just a first step. (8) and (9) will only be useful if one can link the measure theoretic properties of (μ_y) with topological and convergence properties of the operators. But until recently this was done only for integral operators and a special kind of operators with an atomic representation, namely Rieszhomomorphisms (2). Starting from these well known cases we describe now some characterizations of T^s, T^d and T^a obtained in [48], [50].

Let T always be a bounded operator $T : L_1(X,\mathcal{A},\mu) \to L_1(Y,\mathcal{B},\nu)$ where $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ are standard measure spaces with finite μ and ν.

For integral operators there is the classical result of Dunford and Pettis.

THEOREM 3 ([10]). *T is an integral operator if and only if every $A \in \mathcal{A}$, $\mu(A) > 0$, contains some $B \in \mathcal{A}$, $\mu(B) > 0$, such that $T\chi_B$ is (weakly) compact.*

If we replace the bands $L_1(B)$ in this statement by sublattices $L_1(\Sigma)$, then we obtain a characterization of the much larger class B^d.

THEOREM 4 ([50] Th. 4.2). *T has a diffuse representation if and only if every $A \in \mathcal{A}$, $\mu(A) > 0$, contains a non-atomic σ-subalgebra $\Sigma \subset \mathcal{A}$ such that $T|_{L_1(\Sigma,\mu)}$ is (weakly) compact.*

To motivate the next results, recall that T is a Rieszhomomorphism if and only if the images of disjoint functions are still disjoint. Since sequences converging to zero in measure ($f_n \xrightarrow{\mu} 0$) contain subsequences of functions with 'almost' disjoint support the following conditions (11) and (12) on convergence in measure are similar in kind but weaker than this disjointness preserving property.

THEOREM 5 ([50] Th. 6.5). *T has an atomic representation if and only if for all $\varepsilon > 0$ there is an $A \in \mathcal{A}$, $\mu(A^c) \leq \varepsilon$, such that for $T\chi_A$ we have:*

(11) *If (f_n) is bounded in L_1 and $f_n \xrightarrow{\mu} 0$ then $(T\chi_{A_n}) f_n \xrightarrow{\nu} 0$.*

For the next statement, assume that A is generated by a metric d on X and denote by d(A) the diameter of a set $A \subset X$.

THEOREM 6 ([50] *Th. 5.3*). *T has a singular representation if and only if for all* $\varepsilon > 0$ *there is an* $A \in A$, $\mu(A^c) \leq \varepsilon$, *such that for* $T\chi_A$ *we have:*

(12) *If* (f_n) *is bounded in* L_1 *and* $d(\{f_n \neq 0\}) \to 0$ *then* $(T\chi_A)f_n \xrightarrow{\nu} 0$.

There are various other characterizations of T^d, T^a and T^s contained in [48], [50] E.g. one can replace the condition "$d(\{f_n \neq 0\}) \to 0$" in (12) by a more complicated but more natural condition (called 'universal convergence' in [50]) and there is also a characterization of T^d using an equi-integrability notion. Furthermore, T^d, T^a, T^s may be characterized in terms of the vector-valued martingale representing them (see [50]) or by topological conditions on the "kernel" function $y \in Y \to \mu_y \in M(X)$ (cf. [54]).
The idea behind the proof of theorem 6 is related to the following result of Kadec and Pelczynski ([20]): for every bounded sequence $(f_n) \subset L_1$ there is a subsequence (n_k) and sequences (h_k), $(g_k) \subset L_1$ such that $f_{n_k} = h_k + g_k$, $h_k \xrightarrow{\mu} 0$ and (g_k) is equi-integrable. Theorem 4 and 5 essentially follow from (10) and the following purely measure theoretic lemma.

LEMMA 1 ([49]). *For every stochastic kernel* $(\mu_y)_{y \in Y}$ *on* (X, A, μ) *with* μ_y *diffuse and* $\sup_{y \in Y} \|\mu_y\| < \infty$ *there is a* σ*-subalgebra* $\Sigma \subset A$ *such that*

i) $\mu_{|\Sigma} \neq 0$ *and diffuse*

ii) ν*-almost all* μ_y *are* μ*-absolutely continuous on* Σ.

From the definitions of B^i, B^s, B^a and B^d and the above characterizations one easily obtains the following order theoretical and algebraical properties of these classes.

COROLLARY 1 ([42], [50]). B^i, B^s, B^a *and* B^d *are projection bands in the lattice* B *of all bounded linear* L_1*-operators and we have the (lattice) orthogonal decompositions*

$$B = B^i \oplus B^s, \quad B = B^d \oplus B^a.$$

In particular, B^a is the band generated by all Rieszhomomorphisms (2) (see [50], 6.3).

COROLLARY 2. *B^i is a closed two-sided ideal in B; B^d is a closed left (but not a right) ideal. B^a is a closed subalgebra of B.*

The fact that B^s is not a subalgebra of B has received some attention in Harmonic Analysis; we will touch on this in Sec. 6.

3. L_1-SUBSPACES OF BANACH LATTICES AND THE RADON NIKODYM PROPERTY

From the results of Section 2 one can easily derive some interesting characterizations of Banach lattices without L_1-subspaces or with the Radon Nikodym property.

Since c_0 does not have the Radon Nikodym property we can restrict ourselves to Banach lattices not containing c_0. By a well known representation theorem (e.g. [29] 1b.14) there is no loss of generality in assuming that E is a Banach function space on a standard measure space (Y,ν) with

$$L_\infty(Y,\nu) \subset E \subset L_1(Y,\nu) \tag{13}$$

and order-continuous norm. Furthermore, every operator $T : L_1(X,\mu) \to E$ has a representation (5) by a stochastic kernel. Then, by translating the usual Dunford-Pettis representation into our context, we can say that E has the Radon Nikodym property (RNP) if and only if every operator $T : L_1(X,\mu) \to E$ is an integral operator (1). The following characterization of RNP is similar in spirit to the dentability condition for general Banach spaces.

THEOREM 7 ([6]). *For a Banach lattice E as in (13) the following are equivalent*

a) *E has the Radon Nikodym property.*
b) *Every bounded linear operator $T : L_1(X,\mu) \to E$ is an integral operator.*
c) *Every closed, bounded, convex subset D of E is order-dentable, i.e. for some $n \in \mathbb{N}$ we have*

$$D \underset{\neq}{\supset} \overline{\mathrm{conv}}\ \{f \in D : \|f \wedge 1\| \le \tfrac{1}{n}\} .$$

Since 1 is a quasi-interior point of E, $\|f \wedge 1\|$ is small, typically when f is a function which is very small outside a small subset of Y. Hence D is order-dentable if it is not the closed convex hull of its very 'peaky' members.

SKETCH OF PROOF: a) $\Rightarrow$ c) is a slight modification of a standard argument in RNP-theory (see [8]). The more difficult part of the argument in [6] is c) $\Rightarrow$ b). But this also follows from Theorem 6. Assume there is a non-integral operator $T : L_1(X,\mu) \to E$. By decomposing this operator if necessary we may assume that T has a singular representation. Choose an $A \subset X$ as in Theorem 6 and then sets $A_i^n \subset A$, $i = 1,\dots,2^n$, $n \in \mathbb{N}$ such that

i) $A_i^n \cap A_j^n = \phi$ for $i \neq j$, $\bigcup_{i=1}^{2^n} A_i^n = A$

ii) $\max_{i=1}^{2^n} d(A_i^n) \to 0$

By (12) we have $f_i^n := T(\mu(A_i^n)^{-1}\chi_{A_i^n}) \xrightarrow{\mu} 0$. Since the norm of E is order continuous this implies

$$(14) \qquad \max_{i=1}^{2^n} \|f_i^n \wedge 1\| \to 0 \qquad \text{for } n \to \infty.$$

On the other hand, since every $L_1(X,\mu)$-function can be approximated by step-functions $\sum_{i=1}^{2^m} \alpha_i \chi_{A_i^m}$, m large enough, we have

$$(15) \qquad \begin{aligned} T(U_{L_1}) &= T(\overline{\text{conv}}\ \{\mu(A_i^m)^{-1}\chi_{A_i^m},\ i = 1,\dots,2^m,\ m \geq n\}) \\ &\subset \overline{\text{conv}}\ \{f_i^m, i = 1,\dots,2^m,\ m \geq n\} \end{aligned}$$

for all n. It follows from (14) and (15) that $D = \overline{T(U_{L_1})}$ is not order dentable in E. ◻

The connection between L_1-subspaces of E and operator representations is given by

THEOREM 8. For a Banach lattice E *as in (13) the following are equivalent:*

a) Every operator $T : L_1[0,1] \to E$ *has a diffuse representation.*

b) $L_1[0,1]$ *is not a subspace of* E.

c) L_1 0,1 *is not isomorphic to a sublattice of* E, *more precisely, there is no σ-sub-algebra Σ of B such that* $\mu_{|\Sigma}$ *is diffuse and the norm of* $L_1(Y,\nu)$ *and* E *are equivalent for Σ-measurable functions.*

The equivalence b)⟺c) was shown for subspaces of L_1 by Enflo and Starbird ([11]), for dual Banach lattices by Lotz ([31]), for Banach lattices not containing ℓ_n^∞'s uniformly by Johnson, Maurey, Schechtman and Tzafriri in [19] and in the general case by Kalton ([23]).

SKETCH OF PROOF. a) $\Rightarrow$ b) $T : L_1 \to E$ cannot be an isomorphism onto a subspace of E and at the same time have a diffuse representation, since in the latter case there is always a subspace of L_1 on which T is compact (by Theorem 4).
b) $\Rightarrow$ c) is clear.
c) $\Rightarrow$ a) Assume $T : L_1[0,1] \to E$ is not diffuse. Then the atomic part of T is not zero and there is (compare (10)) some $\varepsilon > 0$ such that the set $A = \{\varepsilon < |a_1| < \frac{1}{\varepsilon}\}$ has non-zero ν-measure. Define

$$Sf(x) = \begin{cases} a_1(x)f(\sigma_1(x)) & \text{for } x \in A \\ 0 & \text{for } x \notin A. \end{cases}$$

Since $|S|f \leq |T|f$ for $f \geq 0$ it follows that S maps L_1 into E. Then the image measure $\mu \circ \sigma_1^{-1}$ is ν-absolutely continuous and there is a subset $A_0 \subset A$, $\nu(A_0) > 0$, where the Radon-Nikodym derivative $\frac{d\mu \circ \sigma_1^{-1}}{d\nu}$ is bounded and bounded away from zero. For $\Sigma = \sigma^{-1}(A) \cap A_0$ it is easy to see now, that the norms of E and $L_1(Y,\nu)$ are equivalent on $L_\infty(\Sigma,\nu)$. □

Comparing the proof of Theorem 7 with c) of Theorem 8 one might ask if a Banach lattice without L_1- and c_0-subspaces has already the Radon-Nikodym property. This is true for dual Banach lattices (see [31]) but not in general (see [44]).

4. APPROXIMATION BY WEAKLY COMPACT OPERATORS

In recent years there has been some interest in approximation theory in the question if there is for every bounded linear operator an element of best approximation in the class of compact operators. The answer is 'yes' for bounded linear operators on ℓ_p, $1 \leq p < \infty$, (see [14 Chap. II, Sec. 7], [17, 33, 2] for different methods of proof) and for reasonable integral operators on $L_p[0,1]$, $1 < p < \infty$, but on $L_1[0,1]$ there is even an integral operator without an element of best approximation in the class of compact operators (cf. [47]). However, many of the natural properties of the ideal of compact operators on ℓ_p (especially with respect to spectral theory and perturbation theory, see [38] § 26.6, 26.7 and Sec. 5) belong on $L_1[0,1]$ to the ideal of weakly compact operators and therefore one might consider approximations by weakly compact operators.

Let (X,A,μ) be a standard measure space with finite μ and denote by $\mathcal{B}(L_1)$ and $\mathcal{W}(L_1)$ the class of all bounded linear and of all weakly compact operators on $L_1(X,A,\mu)$, respectively.

THEOREM 9 ([51]). *For every* $T \in \mathcal{B}(L_1)$ *there is an* $S_0 \in \mathcal{W}(L_1)$ *such that*

(16) $$\|T-S_0\| = \inf\{ \|T-S\| : S \in \mathcal{W}(L_1)\} .$$

The proof of Theorem 9 depends on a formula for $\inf\{\|T-S\| : S \in \mathcal{W}\}$ which will also be useful in the next sections. Recall that $T \in \mathcal{B}(L_1)$ is weakly compact if and only if $T(U_{L_1})$ is equi-integrable, i.e. if and only if

(17) $$\Delta(T) = \overline{\lim_{\mu(A)\to 0}} \|\chi_A T\|$$

is zero. In particular, integral operators with an uniformly bounded kernel are weakly compact.

For an integral operator T with kernel k (7) gives

(18) $$\|T\| = \sup_x \int |k(y,x)|dy$$

and we obtain

$$\inf\{ \|T-S\| : S \in \mathcal{W}(L_1)\} \leq \inf_n \|T_n\| \leq \overline{\lim_{\mu(A)\to 0}} \sup_x \int_A |k(y,x)|dy$$

$$= \Delta(T) \leq \inf\{\Delta(T+S) : S \in \mathcal{W}\} \leq \inf\{ \|T-S\| : S \in \mathcal{W}\}$$

where T_n is the integral operator with kernel

(19) $$k_n(y,x) = \begin{cases} k(y,x) & \text{if } |k(y,x)| \geq n \\ 0 & \text{otherwise} \end{cases} .$$

On the other hand, if T is a Riesz homomorphism

$$Tf(y) = a(y)f(\sigma(y))$$

we may choose $A_n \in \mathcal{A}$, $\mu(A_n) \to 0$, such that for $f_n = \mu(A_n)^{-1}\chi_{A_n}$ we have $\|Tf_n\| \to \|T\|$. (Observe that U_{L_1} is the closed absolutely convex hull of all $\mu(A)^{-1}\chi_A$, $\mu(A) \leq \delta$, for every fixed $\delta > 0$.) Since Tf_n is supported by $B_n = \sigma^{-1}(A_n)$, $\mu(B_n) \to 0$, it follows that $\|T\| = \Delta(T)$.
Furthermore, since every weakly compact operator S maps (f_n) into an equi-integrable sequence (Sf_n) and $Tf_n \xrightarrow{\mu} 0$ we have

$$\|T\| \leq \|T+S\| \qquad \text{for all } S \in \mathcal{W},$$

or $\|T\| = \inf\{ \|T-S\| : S \in \mathcal{W}\} = \Delta(T)$.
In the general case, the decomposition of T into its integral part T^i and its singular part T^s and Theorem 6 give

THEOREM 10 ([51]). *For arbitrary* $T \in \mathcal{B}(L_1)$ *we have*

$$\Delta(T) = \inf\{\|T-S\| : S \in \mathcal{W}(L_1)\} = \inf_n \|T^s+(T^i)_n\|$$

where $(T^i)_n$ *is defined analogously to* T_n *in (19).*

In particular, if T has a singular representation, then $\|T\| = \Delta(T)$.

PROOF OF THEOREM 9. If $(\nu_x)_{x\in X}$ represents $T' : L_\infty(X,\mu) \to L_\infty(X,\mu)$ then (7) and Theorem 9 give the formula

$$\Delta(T) = \inf_n \sup_{x\in X} \{\|\nu_x^s\| + \int |k_n(y,x)|d\mu(y)\} \tag{20}$$

where ν_x^s is the μ-singular part of ν_x and k_n is defined as in (19) if k is the kernel of T^i. For all

$$x \in A := \{x \in X : \|\nu_x\| > \Delta(T)\}$$

we choose $t_x \in \mathbb{R}^+$ in such a way that for

$$B_x = \{y : |k(y,x)| > t_x\}$$

we have

$$\|\nu_x^s\| + \int_{B_x} |k(y,x)|d\mu(y) = \Delta(T).$$

Using (20) again one may check that the integral operator S_0 with kernel

$$k_0(y,x) = \begin{cases} k(y,x) & \text{for } y \notin B_x, \quad x \in A \\ 0 & \text{otherwise} \end{cases}$$

satisfies $\|T-S_0\| = \Delta(T)$ and $\Delta(S_0) = 0$.

□

One might ask if $\mathcal{W}(L_1)$ is even an M-ideal in $\mathcal{B}(L_1)$. This would imply (see [17]), that for every $T \in \mathcal{B}(L_1)-\mathcal{W}(L_1)$ there are infinitely many elements of best approximation. In sharp contrast to this the next proposition shows that for every Markovian operator with a singular representation 0 is the only element of best approximation in $\mathcal{W}$.

PROPOSITION 1 ([51]). *Let* $S, T \in \mathcal{B}(L_1)$ *with* $T'1 = \|T\|\,1$.

If either

a) S *has an integral representation and* T *has a singular representation, or*

b) S *has a diffuse representation and* T *has an atomic representation*

then $\|T+S\| = \|T\| + \|S\|$.

The special case, where S is a finite dimensional projection and T = Id follows from [39], Sec. 3, and the case where S is compact and T = Id is treated in [3].

PROOF: Let T' and S' be represented by $(\nu_x)_{x\in X}$ and $(\mu_x)_{x\in X}$, respectively. By assumption we have

$$|\lambda_x| \wedge |\nu_x| = 0, \qquad \|\nu_x\| = |T'|1(x) = \|T\| .$$

Now the claim follows from (7) and

$$\|\lambda_x+\nu_x\| = \|\lambda_x\| + \|\nu_x\| = \|\lambda_x\| + \|T\| .$$

□

5. THE ESSENTIAL SPECTRUM OF L_1-OPERATORS

It follows from results of Kato ([25]) and Pelczynski ([37]) that the ideal $\mathcal{W}(L_1)$ of weakly compact operators in $L_1(X,\mu)$ (we continue with the notation of Sec. 4) is the largest ideal of Riesz operators - i.e. operators with the 'same' spectral theory as compact operators - and also the largest ideal of admissible perturbations of Fredholm operators (s. e.g. [38] § 26.6, 26.7). In particular

$$\sigma_{ess}(S+T) = \sigma_{ess}(T) \quad \text{if} \quad S\in \mathcal{W}(L_1). \tag{21}$$

Put another way, (21) says that the essential spectrum $\sigma_{ess}(T)$ of T equals the spectrum $\sigma(\hat{T})$ of the equivalence class $\hat{T}$ of T modulo $\mathcal{W}(L_1)$:

$$\sigma_{ess}(T) = \sigma(\hat{T}), \quad \hat{T}\in {}^{\mathcal{B}(L_1)}/_{\mathcal{W}(L_1)}. \tag{22}$$

On the other hand it follows from Theorem 9 that

$$\|\hat{T}\|_{\mathcal{B}/\mathcal{W}} = \Delta(T) = \overline{\lim_{\mu(A)\to 0}} \|\chi_A T\| \tag{23}$$

and this indicates that $\Delta(T)$ plays the same role with respect to the essential spectrum as the operator norm does with respect to the whole spectrum. For example, (23) implies that $\Delta(T)$ is a multiplicative semi-norm on $\mathcal{B}(L_1)$ and from (22) and Theorem 9 we get the following formula for the radius $r_{ess}(T)$ of $\sigma_{ess}(T)$.

COROLLARY 3. For all $T\in \mathcal{B}(L_1)$ *we have*

$$r_{ess}(T) = \lim_{n\to\infty} \Delta(T^n)^{1/n}. \tag{24}$$

THE DOEBLIN CONDITION.

As an illustration for formula (24) we mention that for a positive $T\in \mathcal{B}(L_1)$ the condition

$$(25) \qquad \exists\, n \in \mathbb{N} \quad \text{with} \quad \Delta(T^n) < 1$$

is connected with the classical Doeblin condition which is of interest in the theory of Markov processes as a criterion for when the uniform ergodic theorem holds. Let $(\nu_x^m)_{x \in X}$ be a stochastic kernel representing $(T')^m : L_\infty(X,\mu) \to L_\infty(X,\mu)$. Then the Doeblin condition says that there are

$$(26) \qquad n \in \mathbb{N},\ \delta > 0,\ \varepsilon > 0 \ \text{ such that } \mu(A) < \delta \Rightarrow \ \nu_x^n(A) \le 1-\varepsilon \quad \text{a.e.}$$

If one thinks of $(\nu_x^n)_{x \in X}$ as the n-step transition probability of a Markov chain, then (26) says - very roughly speaking - that no matter where you start, you cannot be sure to reach A in n steps if A is just small enough. Since

$$\sup_{x \in X} \nu_x^n(A) = \|(T')^n \chi_A\|_{L_\infty} = \|\chi_A T^n\|_{L_1}$$

(25) and (26) are equivalent. On the other hand, (25) is equivalent to $r_{ess}(T) < 1$ and therefore (25) holds if and only if T is quasi-compact, i.e. there is some $n \in \mathbb{N}$ and a compact $S \in \mathcal{B}(L_1)$ with $\|T^n - S\| < 1$. This gives the well known fact that the Doeblin-condition holds if and only if the corresponding Markov operator is quasi-compact (see e.g. [40] Chap. 6, § 3, [34] Sec. 5.3).

□

Formula (24) implies that if T^n has a singular representation for all $n \in \mathbb{N}$ then $r_{ess}(T)$ equals the usual spectral radius of T. But more information on the spectra can be drawn from the representation (5).

THEOREM 11. a) *If* T^n *has a singular representation for all* $n \in \mathbb{N}$ *then the unbounded components of* $\mathbb{C} - \sigma(T)$ *and* $\mathbb{C} - \sigma_{ess}(T)$ *coincide.*
b) *If* T *has an atomic representation then* $\sigma(T) = \sigma_{ess}(T)$.

Some special cases of b), e.g. for multiplication operators where already observed e.g. in [35].

PROOF: a) Denote by D the unbounded component of $\mathbb{C} - \sigma(T)$. For $\lambda \in \mathbb{C}$ with $|\lambda| > \sigma(T)$ the resolvent $R(\lambda) := (\lambda Id - T)^{-1}$ has the form

$$(27) \qquad R(\lambda) = \sum_{i=0}^{\infty} \lambda^{-i-1} T^i$$

and for all $\lambda \in D$ we have in a small neighbourhood of λ

$$(28) \qquad R(\mu) = \sum_{i=0}^{\infty} (\lambda-\mu)^i \, R(\lambda)^{i+1}.$$

The set $\{\lambda \in D : R(\lambda)^n$ has a singular representation for all $n \in \mathbb{N}\}$ equals D since

it is non-empty (by (27) and the assumption), closed (by Corollary 1) and open by (28).
Assume now that λ_0 belongs to the unbounded component of $\mathbb{C} - \sigma_{ess}(T)$ but not to D. Then λ_0 is an isolated pole of $R(\lambda)$ and there is a circle $\Gamma \subset D$ with midpoint λ_0 such that the spectral projection

$$(29) \qquad P = \frac{1}{2\pi i} \int_\Gamma R(\lambda) d\lambda$$

is finite-dimensional. Since all $R(\lambda)$, $\lambda \in \Gamma$ have a singular representation, P also has a singular representation. But this is impossible for a finite-dimensional operator.

b) follows from the fact that a Fredholm operator with an atomic representation is already invertible. Indeed, assume there is an operator S and finite dimensional operators K_1, K_2 such that

$$(30) \qquad Id + K_1 = TS = TS^a + TS^d$$

$$(31) \qquad Id + K_2 = TS = S^aT + S^dT.$$

By Corollary 2 TS^a has an atomic and TS^d has a diffuse representation. Then (30) implies $Id = TS^a$ and $K_1 = TS^d$. In particular it follows that S^d is finite-dimensional (since T is a Fredholm operator) and (31) implies $Id = S^aT$. Hence T is invertible.
□

To give an illustration for the use of the representation (5) and of Δ in spectral theory we consider a linear transport equation:

$$(32) \qquad \frac{\partial F}{\partial t}(t,x,v) = -\Sigma v_i \frac{\partial}{\partial x_i} F(t,x,v) - \sigma(x,v)F(t,x,v) + \int_V k(x,v,v')F(t,x,v')dv' .$$

Here, $x \in D$ and $v \in V$ where D (the 'position space') is open, convex and bounded in $\mathbb{R}^3$ and V (the 'velocity space') is compact in $\mathbb{R}^3$. If

$$(33) \qquad \begin{aligned} A\, f(x,v) &:= -\Sigma v_i \frac{\partial}{\partial x_i} f \\ M\, f(x,v) &:= \sigma(x,v) f(x,v), \quad 0 \leq \sigma \in L_\infty(D \times V) \\ K\, f(x,v) &:= \int_V k(x,v,v') f(x,v')dv', \quad 0 \leq k \ \text{ with} \end{aligned}$$

$$(34) \qquad \sup_{x \in D, v' \in V} \int k(x,v,v')dv < \infty$$

then the 'absorption coefficient' σ and the 'scattering kernel' k define bounded linear operators M and K in $L_1(D \times V)$ and the linear Boltzmann operator $B = A-M+K$ is an unbounded generator of a strongly continuous semi-group S_t in $L_1(D \times V)$.

Then for an appropriate initial distribution $f_0 \in D(B)$ the solution of (32) is given by

$$F(t,x,v) = S_t f(x,v).$$

This indicates that the asymptotic behavior of the solutions for large times is related to the spectra of the operators S_t (see [24] Chap. 11 for a more detailed discussion). Here we concentrate on $r_{ess}(S_t)$. Since S_t is not known explicitely we apply a perturbation argument following [45] and [46]: The 'streaming' operator A-M generates a well known semi-group, namely

$$T_t\, f(x,v) = \chi_D(x-vt)\exp[-\int_0^t \sigma(x-vs,v)ds]f(x-vt,v)$$

whose essential spectrum can be determined in concrete cases. For example, if V is a ball in $\mathbb{R}^3$, then

$$\sigma(T_t) = \{\lambda \in \mathbb{C} : |\lambda| \le e^{-\lambda^* t}\}, \quad \lambda^* = \lim_{n\to\infty} \text{ess inf}\{\sigma(x,v) : x \in D, \|v\| \le \tfrac{1}{n}\}.$$

Since T_t has an atomic representation we have by theorem 11 that $\sigma(T_t) = \sigma_{ess}(T_t)$ and it remains to show that

$$(35) \qquad r_{ess}(T_t) = r_{ess}(S_t) \qquad \text{for} \quad t > 0.$$

The semi-groups S_t and T_t are related by Duhamel's formula

$$(36) \qquad S_t = T_t + \int_0^t T_s K S_{t-s} ds$$

but (21) cannot be applied yet because the last operator in (36) is not weakly compact in general. The main reason is that $K : L_1(D \times V) \to L_1(D \times V)$ - although occasionally called an 'integral operator' in the literature - is really a good example for an operator with a diffuse but singular representation, namely we have

$$K\, f(x,v) = \int f d\mu_{(x,v)}, \qquad \mu_{(x,v)} = \delta_x \otimes k(x,v,\cdot)d\lambda$$

where λ is the Lebesgue-measure on V (not on $D \times V$). By iterating formula (36) once we obtain

$$S_t = T_t + \int_0^t T_s K T_{t-s} ds + \int_{\substack{0 \le s_1, s_2 \\ s_1+s_2 \le t}} T_{s_1} K T_{s_2} K S_{t-s_1-s_2} ds_1 ds_2$$

Since Δ is a multiplicative semi-norm on $\mathcal{B}(L_1)$ and since

$$(37) \qquad \Delta(\int_0^t U_s ds) \le \int_0^t \Delta(U_s) ds$$

for every strongly continuous $s \to U_s \in \mathcal{B}(L_1)$ (see [53]), we obtain

$$(38)\qquad \Delta(S_t) \le \Delta(T_t) + \int_0^t \Delta(T_s)\Delta(K)\Delta(T_{t-s})ds + \int_{\substack{0\le s_1,s_2\le t\\ s_1+s_2\le t}} \Delta(T_{s_1})\Delta(KT_{s_2}K)\Delta(S_{t-s_1-s_2})ds_1ds_2 .$$

To estimate the first integral observe that by (24)

$$r_{ess}(T_t) = \lim_{n\to\infty} \Delta(T_{nt})^{1/n} = \exp(\lim_{n\to\infty} \frac{1}{n} \log \Delta(T_{nt})), \text{ or}$$

$$(39)\qquad r_{ess}(T_t) = e^{\omega_T t} \qquad \text{with} \qquad \omega_T := \lim_{t\to\infty} t^{-1}\ln \Delta(T_t)$$

i.e. ω_T is an analogue of the 'type' of a semi-group (T_t) with respect to the essential spectral radius. For $\omega_T < \omega' < \omega''$ and some C we have $\Delta(T_t) \le Ce^{t\omega'}$ and if we assume that

$$(40)\qquad \Delta(KT_s K) = 0 \qquad \text{for all } 0 < s$$

then for large t (38) becomes

$$\Delta(S_t) \le e^{t\omega'} + C\Delta(K)te^{t\omega'} \le e^{t\omega''} .$$

Therefore $\omega_S := \lim_{t\to\infty} \frac{1}{n} \log\Delta(S_t) \le \omega''$ for all $\omega'' > \omega_T$ and (39) implies that $r_{ess}(S_t) \le r_{ess}(T_t)$. The inequality $r_{ess}(S_t) \ge r_{ess}(T_t)$ also follows from (39). Indeed, since all S_t, T_t are positive operators we get from (36) that $S_t \ge T_t$ and therefore $\Delta(S_t) \ge \Delta(T_t)$ for all $t \ge 0$.
It remains to give reasonable conditions so that (40) is fullfilled. Using a criterion for conditionally weak compactness of vector-valued functions from [5] one can show

THEOREM 12 ([53]). *If in addition to (33) we assume that the scattering kernel* k *satisfies*

$$(41)\qquad \sup_{x\in D, v'\in V} \int_E k(x,v,v')dv \to 0 \qquad \textit{for} \quad \mu(E) \to 0$$

then $\quad r_{ess}(S_t) = r_{ess}(T_t) = r(T_t) \quad$ *for* $\quad t \ge 0$.

If one interprets (32) as the neutron transport equation, then $k(x,\cdot,v')$ corresponds to the velocity distribution of a particle that emerge from a collision between a particle with the velocity v' and a host particle at the point x of the position space. Hence condition (41) says that all these distributions form an equi-integrable subset of $L_1(V)$. Theorem 12 was known so far under stronger restrictions on k, e.g.

- (Greiner, [16]) k uniformly bounded,
- (Voigt, [46]) $k(x,v,v') \leq g(v)$ for all $x \in D$, $v' \in V$ and some fixed $g \in L_1(V)$.

Versions of theorem 12 for unbounded D and V are contained in [46] and [53].

6. CONVOLUTION OPERATORS

In this section we show that for some non trivial results of harmonic analysis one can give rather short functional analytic proofs using the representation (5). So, let λ be the right Haar measure of a locally compact group G with a countable base of its topology (then G is homeomorphic to a complete metric space and the results of the previous sections apply with slight changes).
The first observation is close to some results of Oberlin (e.g. see [36]).

THEOREM 13. *If* $T : L_1(G,\lambda) \to L_1(G,\lambda)$ *is translationinvariant, bounded and continuous with respect to convergence in measure, then* T *is an infinite sum of translation operators, i.e. there are* $x_i \in G$, $\alpha_i \in \mathbb{R}$ *such that* $\Sigma|\alpha_i| < \infty$ *and*

$$T = \sum_{i=1}^{\infty} \alpha_i T_{x_i} \qquad , \qquad T_{x_i}(f)(y) = f(y-x_i).$$

PROOF: It follows from the representation theorem 1 and the translation invariance that $Tf = f * \mu$ for some measure μ on G. By theorem 5 T has an atomic representation and so, by (10), there are $x_i \in G$, $a_i \in \mathbb{R}$ with $\Sigma|a_i| < \infty$ and $\mu = \Sigma a_i \delta_{x_i}$.
It follows now that

$$f * \mu(x) = \Sigma a_i f(x-x_i) \qquad \text{a.e.}$$

□

Consider M(G) as a Banachalgebra with respect to convolution of measures. It is well known that the subset of all $\mu \in M(G)$ which are singular with respect to the Haar measure λ on G is not a subalgebra but there is a stronger result in this direction.

THEOREM 14. *If* $\nu \in M(G)$ *and the convolution product* $\nu * \mu$ *is* λ*-singular for all* λ*-singular* $\mu \in M(G)$, *then* λ *is an atomic measure.*

This is due to Doss ([9]) in the case of abelian groups (see also [15] Sec. 7.5). A proof similar to the following one was found independently by M. Talagrand ([43]).

PROOF ([48] p. 77): Assume to the contrary that the diffuse part ν_0 of ν is non-zero. Then ν_0 defines a convolution operator $T(\mu) := \mu * \nu_0$ in $L_1(G)$ and in $M(G)$ with a diffuse representation. By Theorem 4 there is a σ-subalgebra Σ of Borel sets of G, such that $\lambda_{|\Sigma}$ is diffuse and $T_{|L_1(\Sigma,\lambda)}$ is compact. Choose a sequence $f_n \in L_1(\Sigma,\lambda)$, $\|f_n\| = 1$, such that $f_n \cdot \lambda$ converges weakly (in the $\sigma(M(G), C(G))$-topology) to some measure μ_0 which is singular with respect to λ. Since $T : M(G) \to M(G)$ is weakly continuous and also compact on $L_1(\Sigma,\lambda)$ it follows that $(Tf_n)\lambda$ converges in the norm topology to $T(\mu_0) = \mu_0 * \nu$. But $L_1(G,\lambda)$ is norm closed and we obtain that $\nu_0 * \mu_0$ is λ-absolutely continuous. This contradicts the assumption, that $|\mu_0 * \nu| \geq |\mu_0 * \nu_0|$ is λ-singular. □

In [56] Zafran discusses a class of measures whose spectrum with respect to M(G) is essentially given by the Fourier transform. Let G now be a compact group with dual group Γ and denote by C the class of all measures μ whose Fourier transforms vanish at ∞ and with

$$\sigma(\mu|M(G)) = \hat{\mu}(\Gamma) \cup \{0\}.$$

One of the main results of [56] is

THEOREM 15. C is an L-ideal, i.e. an algebraic ideal and a sublattice of M(G).

Theorem 15 can be derived easily from the following characterization of C contained in [4] and for which we will give a short proof using formula (24).

PROPOSITION 2. $\mu \in M(G)$ *belongs to C if and only if* $\|(\mu^n)^s\|^{1/n} \to 0$ *for* $n \to \infty$ *where* $(\mu^n)^s$ *is the λ-singular part of the* n*-fold convolution product* μ^n.

(To derive theorem 15 observe that
$$((\mu*\nu)^n)^s \leq (\mu^n)^s * \nu^n.)$$

PROOF of Proposition 2: It is well known that $\sigma(\mu|M(G))$ equals the spectrum $\sigma(T|B(L_1))$ of the convolution operator $Tf = f * \mu$ and that the Fourier transform of μ gives the eigenvalues of T. Therefore, $\mu \in C$ if and only if $\sigma_{ess}(T) = \{0\}$ or $\Delta(T^n)^{1/n} \to 0$ by (24). The integral part of T^n is the weakly compact convolution operator $(T^n)^i f = f * (\mu^n)^i$ where $(\mu^n)^i$ is the integral part of μ^n. By theorem 10 we have

$$\Delta(T^n) = \Delta((T^n)^s) = \|(T^n)^s\| = \|(\mu^n)^s\|$$

and the result follows. ■

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TAUBERIAN THEOREMS FOR OPERATORS ON L^∞ AND SIMILAR SPACES

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In this paper we prove some uniform convergence theorems for operators on a Grothendieck space with the Dunford-Pettis property. As consequences we obtain 1) that on such spaces (in particular on L^∞ and $H^\infty(D)$) every C_0-semigroup is uniformly continous, 2) that on such spaces the strong ergodic theorem becomes a uniform ergodic theorem, and 3) Dean's result that such a space does not have a Schauder decomposition.

1. INTRODUCTION

It is well known that L^∞ differs greatly in some respects from the spaces L^p for $1 \leq p < \infty$. On the other hand, L^∞ shares with L^1 the Dunford-Pettis property; and with L^p, $1 < p < \infty$, the property of being a Grothendieck space. For a more detailed discussion we refer to Section 2, where we give other examples of spaces with these two properties. Here let us mention that recently Bourgain proved that $H^\infty(D)$ is a Grothendieck space with the Dunford-Pettis property. This was pointed out at the Conference by Professor A. Pełczyński.

In this article we investigate the uniform convergence of bounded linear operators on L^∞ and similar spaces, or to be more precise, on Grothendieck spaces with the Dunford-Pettis property. It turns out that in many interesting cases the strong convergence of operators on such a space implies the uniform convergence. Let us at first give three known examples in this direction.

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A sequence (P_n) of bounded linear operators on a Banach space E is called a (weak) Schauder decomposition if the following conditions hold:

(i) $P_m P_n = P_{\inf(n,m)}$ for all $n,m \in \mathbb{N}$

(ii) $(P_n x)$ converges (weakly) to x for every $x \in E$

(iii) $P_n \neq P_m$ for $n \neq m$.

Usually, the (weak) Schauder decomposition is defined in terms of closed subspaces. We prefer here the equivalent operator theoretic definition. Now Dean shows in [8] that a Grothendieck space with the Dunford-Pettis property does not have a (weak) Schauder decomposition. This means, if on such a space a sequence of bounded linear operators (P_n) satisfies (i) and (ii) above, then (iii) cannot hold, which implies that $P_n = 1$ for n sufficiently large, or equivalently, since the operators P_n are projections, that $\lim \|P_n - 1\| = 0$. So, trivially here strong convergence implies convergence in the uniform operator topology.

In [14] Kishimoto and Robinson show that a C_0-semigroup (i.e., a strongly continuous semigroup in the sense of [10], VIII.1.1) of positive operators on L^∞ is uniformly continuous and ask whether this is also true without the assumption that the operators are positive. (It seems that this question has also been raised by other authors.)

Let us call a bounded linear operator T on a Banach space strongly (resp. uniformly) ergodic, if the means $T_n = n^{-1} \sum_{i=0}^{n-1} T^i$ converge in the strong (resp. uniform) operator topology. In [2] Ando shows that if T is an irreducible positive contraction on L^∞, then T is strongly ergodic. In [15] we show that under the same hypotheses T must be uniformly ergodic. More generally, it is shown in [15] that an irreducible positive operator T on C(X), where C(X) is a Grothendieck space, is uniformly ergodic iff T is strongly ergodic.

Starting with some problems concerning strongly ergodic operators the author was ultimately led to the results of [17]. There we show that the answer to the question of Kishimoto and Robinson mentioned above is affirmative not only for L^∞, but for every Grothendieck space with the Dunford-Pettis property. Moreover, we show that on such a space every strongly ergodic contraction is necessarily uniformly ergodic and we give a new proof of Dean's result.

Most of the results which we present here are already contained in [17]. But as announced there, in the present article all these results are deduced from two Tauberian theorems from the last section of [17]. These two theorems (Theorems 1 and 2 below) are uniform convergence theorems for (UM)-sequences (to be defined in the next paragraph). These sequences provide a natural frame for many results in ergodic theory, Schauder decompositions, and semigroups (cf. [18]).

Let (S_n) be a sequence of bounded linear operators on a Banach space E. We will call such a sequence a (UM)-sequence if the following conditions hold:

(UM_0) $\quad \sup \|S_n\| < \infty$

(UM_1) $\quad \lim_n \|S_m(S_n - 1)\| = 0$ for every $m \in \mathbb{N}$

(UM_2) $\quad \lim_n \|(S_n - 1)S_m\| = 0$ for every $m \in \mathbb{N}$.

We give several examples of (UM)-sequences, some of which will be used in Section 4 below.

Examples. 1. Let $(P_n) \subset \mathcal{L}(E)$ be uniformly bounded. If (P_n) satisfies (i) above, then (P_n) is a (UM)-sequence. In particular, every (weak) Schauder decomposition (P_n) is a (UM)-sequence.

2. Let (e_i) be a basis of the infinite dimensional Banach space E. Let $P_n x = \sum_{i=0}^{n} \alpha_i e_i$ for $x = \sum_{i=0}^{\infty} \alpha_i e_i$. Then (P_n) is a Schauder decomposition and so a (UM)-sequence.

3. Let (S, Σ, μ) be a probability space, let (Σ_n) be an increasing sequence of σ-subalgebras of Σ, and let $E = L^p(S, \Sigma, \mu)$, $1 \leq p \leq \infty$. If $P_n f$, $f \in E$, is the conditional expectation of f with respect to Σ_n, then (P_n) is a (UM)-sequence. If, in addition, Σ is the σ-algebra generated by the Σ_n, $1 \leq p < \infty$, and $\Sigma_n \neq \Sigma_{n+1}$ for all n, then (P_n) is a Schauder decomposition.

4. Let D be a non-empty subset of the complex plane and let $R: D \to \mathcal{L}(E)$ satisfy the resolvent equation

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu)$$

for all $\lambda,\mu \in D$. If $(\lambda_n) \subset D$ with $\lim |\lambda_n| = \infty$, and if the operators $\lambda_n R(\lambda_n)$ are uniformly bounded, then (S_n) with $S_n = \lambda_n R(\lambda_n)$ is a (UM)-sequence. Similarly, if $(\lambda_n) \subset D$ with $\lim \lambda_n = \alpha \in \mathbb{C}$ and the operators $(\lambda_n - \alpha)R(\lambda_n)$ are uniformly bounded, then (S_n) with $S_n = 1 - (\lambda_n - \alpha)R(\lambda_n)$ is a (UM)-sequence.

5. Let T be a bounded linear operator on a Banach space E. If the means $(\sum_{i=0}^{n-1} T^i)/n$ are uniformly bounded and $\lim \|T^n\|/n = 0$, then (S_n), where $S_n = 1 - T_n$, is a (UM)-sequence.

6. Let $\{T(t) : 0 \leq t\}$ be a C_0-semigroup of operators on a Banach space E. Let $S_n = \int_0^1 T(t/n)dt$, where the integral is taken in the strong operator topology. Then (S_n) is a (UM)-sequence. Another (UM)-sequence associated to $\{T(t) : 0 \leq t\}$ is $(\lambda_n(\lambda_n - A)^{-1})$, where A is the infinitesimal generator and $0 \leq \lambda_n \in \rho(A)$ with $\lim \lambda_n = \infty$.

Finally, we want to point out connections between (UM)-sequences and martingales. Let (S_n) be a (UM)-sequence of operators on a Banach space E. A sequence $(x_n) \subset E$ is called an (S_n)-martingale if $\lim_n S_m x_n = x_m$ for all $m \in \mathbb{N}$. Obviously, if (S_n) is as in Example 3, then the (S_n)-martingales are just the martingales in the usual sense. Now let (x_n) be an (S_n)-martingale. It is easily proved (cf. proof of Lemma 1 below) that if (x_n) has a weak cluster point, say x, then (x_n) converges to x and $x_n = S_n x$ for all $n \subset \mathbb{N}$ (actually, this also holds under weaker assumptions on (S_n), we refer to [18] for this and other related facts). We also note that if (S_n) is a (UM)-sequence then (S_n) itself is a $(\overline{S}_n)$-martingale in the Banach space $\mathcal{L}(E)$, where $\overline{S}_n$ denotes left multiplication $T \mapsto S_n T$; similarly, for the right multiplication.

2. GROTHENDIECK SPACES WITH THE DUNFORD-PETTIS PROPERTY

In this section X denotes always a compact Hausdorff space. X is called Stonian (resp. σ-Stonian) if the closure of every open subset (resp. open F_σ-set) of X is open. X is called an F-space if two disjoint open F_σ-sets have disjoint closures. Clearly, every Stonian space is σ-Stonian and every σ-Stonian space is an F-space. For any measure space (S,Σ,μ), localizable or not, we denote the space $L^p(S,\Sigma,\mu)$ ($p = 1$ or $p = \infty$) by L^p. If Σ is an algebra of subsets of a set S, then $B(S,\Sigma)$ denotes the Banach space of all bounded Σ-measurable scalar valued functions on S with the sup-norm.

A Banach space E is said to have the Dunford-Pettis property if $\lim_n \langle x_n, x_n' \rangle = 0$ whenever $(x_n) \subset E$ tends weakly to zero and $(x_n') \subset E'$ tends weakly to zero. If E has the Dunford-Pettis property, then every weakly compact projection on E is of finite rank. Therefore every reflexive Banach space with the Dunford-Pettis property is finite dimensional. The Dunford-Pettis property is inherited by complemented subspaces and by preduals. Grothendieck [11] shows that every space C(X) has the Dunford-Pettis property. This implies the classical result of Dunford and Pettis that every space L^1 has the Dunford-Pettis property. Another consequence is that every AM-space, in particular, every closed ideal of C(X) has the Dunford-Pettis property.

A Banach space E is called a Grothendieck space if every sequence $(x_n') \subset E'$ which converges for the weak* topology to zero converges weakly to zero. If E is a Grothendieck space, then every quotient space of E, in particular, every complemented subspace of E is a Grothendieck space. Obviously, every reflexive Banach space is a Grothendieck space. Every separable quotient space of a Grothendieck space is necessarily reflexive (cf. [11], p. 169). Non-trivial examples of Grothendieck spaces will follow next.

The following Banach spaces are Grothendieck spaces with the Dunford-Pettis property:

1) C(X) where X is Stonian, σ-Stonian, or an F-space.
2) L^∞, in particular, ℓ^∞.
3) $B(S,\Sigma)$ where Σ is a σ-algebra.
4) Injective Banach spaces.
5) The Hardy algebra $H^\infty(D)$ of all bounded analytic functions on the open unit disk.

For the spaces listed above, the results of Sections 3 and 4 can be applied. Let us make some comments about these spaces above.

As we have already mentioned, every space C(X) has the Dunford-Pettis property. Grothendieck [11] proves if X is Stonian then every sequence in the dual of C(X) which converges in the weak* topology also converges in the weak topology. This result was extended by Ando [1] for X σ-Stonian and by Seever [20] for X an F-space. Hence every space under 1) is a Grothendieck space with the Dunford-Pettis property. For other interesting examples of spaces C(X) which are Grothendieck spaces we refer to Talagrand [21] and Haydon [12].

We turn to the spaces listed under 2) and 3). It is well known that these spaces are norm and order isomorphic to spaces C(X) (cf. [10], V.8.11 and p. 716). Since L^∞ and $B(S,\Sigma)$ where Σ is a σ-algebra are σ-order complete, X must be σ-Stonian ([19], II.7.7). Thus it follows from 1) that the spaces in 2) and 3) are Grothendieck spaces with the Dunford-Pettis property.

Every injective Banach space is isomorphic to a complemented subspace of a space C(X) where X is Stonian. So it follows from 1) that the spaces under 4) are Grothendieck spaces with the Dunford-Pettis property.

Now to 5). Bourgain shows in [4] that $H^\infty(D)$ has the Dunford-Pettis property and in [5] that $H^\infty(D)$ is a Grothendieck space. These results depend on deep results due to Bourgain in harmonic analysis (cf. [6] for some of these relevant facts).

We conclude this section with the remark that a Banach space E is a Grothendieck space with the Dunford-Pettis property iff every weak* convergent sequence in E' converges weakly, moreover, uniformly on weakly compact subsets of E, or equivalently, iff every bounded linear map from E into c_0 is weakly compact and maps weakly compact sets into norm compact sets.

3. TAUBERIAN THEOREMS

Now we come to the main results of this article. We will give various conditions which imply that a (UM)-sequence of operators on a Grothendieck space with the Dunford-Pettis property converges in the uniform operator topology. Thus these results are of Tauberian nature (cf. [10], p. 593). For the definition of (UM)-sequences we refer to the Introduction, and for examples of Grothendieck spaces with the Dunford-Pettis property we refer to Section 2.

We start with two lemmata.

Lemma 1. Let (S_n) be a (UM)-sequence of operators on a Banach space E. Let G be the closure of $\cup S_n E$ and let $F = \cap S_n^{-1}\{0\}$. Then the following assertions hold:

(a) G is a linear subspace and thus the weak closure of $\cup S_n E$; moreover, $G = \{x: x = \lim S_n x\}$.

(b) If for every $x \in E$ the sequence $(S_n x)$ has a weak cluster point, then (S_n) converges strongly to a projection P with G as range and F as kernel.

Proof. (a) Let $H = \{x: x = \lim S_n x\}$. Then H is a linear subspace and $H \subset G$. That H is norm closed follows from (UM_0). On the other hand, (UM_2) implies that $\cup S_n E \subset H$. Hence $G = H$. Since G, as a closed linear subspace, is weakly closed, G is also the weak closure of $\cup S_n E$.

(b) It follows from (a) that $F \cap G = \{0\}$. It suffices now to show that $E = F + G$. Then E is the direct sum of F and G and (S_n) tends to the desired projection, since $S_n|_F = 0$ for all n and $(S_n|_G)$ tends strongly to the identity on G by (a). Now let $x \in E$ be given and let z be a weak cluster point of $(S_n x)$. By (a), $z \in G$. Let m be fixed. Since $z - x$ is a weak cluster point of $((S_n - 1)x)$, $S_m(z - x)$ is a weak cluster point of $(S_m(S_n - 1)x)$. But this sequence converges in norm to zero by (UM_1). Hence $S_m(z - x) = 0$, which shows that $z - x \in F$. Therefore, $x = z - (z - x) \in G + F$, and so $E = F + G$.

More results concerning the strong convergence of (UM)-sequences can be found in [18].

The next lemma is a slight variation of Theorem 2 from [17].

Lemma 2. Let (V_n) be a sequence of bounded linear operators on a Banach space E with the Dunford-Pettis property. Suppose that the following two conditions hold:

(i) $(V_n x_n)$ tends weakly to zero whenever $(x_n) \subset E$ is bounded.

(ii) $(V_n' x_n')$ tends weakly to zero whenever $(x_n') \subset E'$ is bounded.

Then $\lim_n \|V_n^2\| = 0$. In particular, $1 + V_n$ is invertible for n sufficiently large.

Proof. We choose a sequence (x_n) in E with $\|x_n\| \leq 2$ for all n and a sequence (x_n') in E' with $\|x_n'\| \leq 1$ for all n such that $\|V_n^2\| = \langle V_n^2 x_n, x_n' \rangle$. By (i) and (ii) the sequences $(V_n x_n)$ and $(V_n' x_n')$ tend to zero for $\sigma(E,E')$ and $\sigma(E',E'')$ respectively. Since E has the Dunford-Pettis property, $\lim \langle V_n x_n, V_n' x_n' \rangle = 0$, and so, $\lim \|V_n^2\| = 0$. Hence, for n sufficiently large, $\|V_n^2\| < 1$ and so $1 - V_n^2$ is invertible. But then $(1 - V_n)(1 - V_n^2)^{-1} = (1 - V_n^2)^{-1}(1 - V_n)$ is the inverse of $1 + V_n$.

We are now ready to prove the first two main results, which also appear as Theorem 10 and Theorem 11 respectively in [17].

Theorem 1. Let E be a Grothendieck space with the Dunford-Pettis property and let (S_n) be a (UM)-sequence of operators on E. If $\cup S_n E$ is weakly dense in E, in particular, if (S_n) converges to the identity in the weak or strong operator topology, then $\lim \|S_n - 1\| = 0$.

Proof. Let $V_n = S_n - 1$. We will show that (V_n) satisfies conditions (i) and (ii) of Lemma 2. Consequently, since E has the Dunford-Pettis property, Lemma 2 implies the existence of a natural number, say m, such that $S_m = 1 + V_m$ is invertible. It will then follow from $\|S_n - 1\| \leq \|S_m^{-1}\| \|S_m(S_n - 1)\|$ and from (UM_2) that $\lim \|S_n - 1\| = 0$.

So suppose that $\cup S_n E$ is weakly dense. Then (a) of Lemma 1 implies that $x = \lim S_n x$ for every $x \in E$. In other words, (V_n) tends in norm to zero for every $x \in E$. It follows that $(V_n' x_n')$ tends to zero for $\sigma(E',E)$ if $(x_n') \subset E'$ is bounded, and since E is a Grothendieck space, $(V_n' x_n')$ tends weakly to zero. Hence (ii) of Lemma 2 is satisfied, and in particular $(V_n' x')$ tends weakly to zero for every $x' \in E'$. This implies that $(S_n' x')$ tends weakly to x' for every $x' \in E'$, and since (S_n') is clearly a (UM)-sequence as well, Lemma 1 shows that (S_n') converges strongly to the identity on E'. As above we conclude that $(V_n x_n)$ tends weakly to zero whenever $(x_n) \subset E$ is bounded, so, condition (i) of Lemma 2 is satisfied. Q.E.D.

Remark 1. A simple modification of the proof of Theorem 1 gives the following result: Let (S_n) be a (UM)-sequence of operators on a Banach space E with the Dunford-Pettis property. If (S_n'') converges weakly to the identity, then (S_n) converges uniformly to the identity.

Theorem 2. Let E be a Grothendieck space with the Dunford-Pettis property and let $(S_n) \subset \mathcal{L}(E)$ be a (UM)-sequence. If for every $x \in E$ the sequence $(S_n x)$ has a weak cluster point, then (S_n) converges uniformly to a projection P with $PE = \overline{\cup S_n E}$ and $(1 - P)E = \cap S_n^{-1}\{0\}$.

Proof. Let F and G be as in Lemma 1. Applying Lemma 1, (b), we see that (S_n) converges strongly to a projection P with PE = G and (1 - P)E = F. Since $S_n G \subset G$ for all $n \in \mathbb{N}$, $(S_n|_G)$ is a (UM)-sequence of operators on G, which converges strongly to the identity on G. Now G, as a complemented subspace of E, is a Grothendieck space with the Dunford-Pettis property. Hence it follows from Theorem 1 that the operators $S_n|_G$ converge uniformly. Since all the S_n vanish on F, it is clear that $\lim\|S_n - P\| = 0$. Q.E.D.

Remark 2. Let (S_n) be a (UM)-sequence of operators on a Banach space E with the Dunford-Pettis property. If (S_n'') converges weakly to an operator whose range is $\sigma(E'',E')$ closed, then the operators S_n converge for the uniform operator topology to a projection with $\overline{\cup S_n E}$ as range and $\cap S_n^{-1}\{0\}$ as kernel. The condition that the range is $\sigma(E'',E')$ closed cannot be dropped as the following example shows: Let (e_i) be the usual basis of ℓ^1 and let (P_n) be as in Example 2 of the Introduction. Then (P_n'') converges strongly, but (P_n) does not converge in the uniform operator topology.

Next we show that if, for a (UM)-sequence of operators (S_n) on a Grothendieck space E with the Dunford-Pettis property, $\overline{\cup S_n E}$ or $\cap S_n^{-1}\{0\}$ is a "large" subspace of E, then this subspace is of finite codimension and (S_n) converges in the uniform operator topology.

Theorem 3. Let E be a Grothendieck space with the Dunford-Pettis property and let $(S_n) \subset \mathcal{L}(E)$ be a (UM)-sequence. If $E/\cap S_n^{-1}\{0\}$ is reflexive, then (S_n)

converges uniformly to a projection of finite rank with $\overline{\cup S_n E}$ as range and $\cap S_n^{-1}\{0\}$ as kernel.

Proof. Let F and G be as in Lemma 1. Let $F_1 = \cap S_n'^{-1}\{0\}$ and let G_1 be the weak* closure of $\cup S_n'E'$. Clearly, (S_n') is a (UM)-sequence. The norm closure of $\cup S_n'E'$ is by Lemma 1 a linear subspace, which implies that G_1 is a linear subspace as well. It is easily checked that G_1 is the annihilator of F and that F_1 is the annihilator of G. By assumption, E/F is reflexive. Therefore its dual, G_1, is reflexive, which implies that on G_1 the weak* and the weak topology coincide. Hence G_1 is the weak closure of $\cup S_n'E'$, and so by Lemma 1 also the norm closure of that set. Hence $F_1 \cap G_1 = 0$. Now let $x' \in E'$ be given and let z' be a $\sigma(E',E)$-cluster point of the bounded sequence $(S_n'x')$. An argument similar to that in the proof of Lemma 1 shows that $x' = z' - (z' - x') \in G_1 + F_1$. Hence E' is the direct sum of the $\sigma(E',E)$-closed subspaces G_1 and F_1. Therefore, E is the direct sum of their annihilators F and G. This implies that (S_n) converges strongly to the projection P with G as range and F as kernel. But by Theorem 2, (S_n) converges in norm. Since E/F is assumed to be reflexive, G is reflexive. Hence P is a weakly compact projection and therefore of finite rank since E has the Dunford-Pettis property. Q.E.D.

Lemma 3. Let X be a topological Hausdorff space and let (x_n) be a relatively compact sequence in X. If the set A of all cluster points of (x_n) is metrizable, then the closure of the set $\{x_n\}$ is metrizable.

Proof. Without loss of generality we may assume that X is the closure of the set $\{x_n\}$, so that X is compact. Let $I = \{f \in C(X) : f(A) = \{0\}\}$. Since $X\setminus A$ is countable, I is finite dimensional or isometric to c_0. Hence I is separable. Since A is a compact metrizable space, C(A) is separable. As C(A) is isometric to C(X)/I, and since the separability of C(X)/I and of I implies the separability of C(X), X is metrizable.

Lemma 4. Let (S_n) be a (UM)-sequence of operators on a Banach space E. If $((S_n - 1)x_n)$ is relatively weakly compact for every bounded sequence $(x_n) \subset E$, then the operators $(1 - S_n)'$ converge strongly to a weakly compact projection.

Proof. Clearly, the assumption implies that $(S_n x)$ is relatively weakly compact for every $x \in E$. It follows from (b) of Lemma 1 that (S_n) converges strongly to a projection P with $PE = \overline{\cup S_n E}$ and $(1 - P)E = \cap S_n^{-1}\{0\}$. Hence $S_n P = S_n = PS_n$ for all $n \in \mathbb{N}$. Now let $(x_n) \subset E$ be bounded. Then $((S_n - 1)Px_n)$ is relatively weakly compact by assumption. We claim that this sequence converges weakly to zero. Indeed, let x be a weak cluster point. Then $S_m x = 0$, since $\lim_n \|S_m(S_n - 1)Px_n\| = 0$ by (UM_2). So $x \in (1 - P)E$. On the other hand, $x \in PE$, since P commutes with all operators $S_n - 1$ and PE is weakly closed. Hence $x = 0$, which shows that $((S_n - 1)Px_n)$ tends weakly to zero whenever $(x_n) \subset E$ is bounded. This, together with the equality $S_n - P = (S_n - 1)P$ for all n, easily implies that (S_n') converges strongly to P', and so $(1 - S_n)'$ converges strongly to the projection $(1 - P)'$. It remains to show that $(1 - P)'$, or equivalently, that $1 - P$ is weakly compact. Let $(x_n) \subset (1 - P)E$ be bounded. Then $S_n x_n = 0$ for all n. Hence $(x_n) = ((1 - S_n)x_n)$ is relatively weakly compact by assumption. Hence $(1 - P)E$ is reflexive and so $(1 - P)$ is weakly compact.

Theorem 4. Let E be a Grothendieck space with the Dunford-Pettis property and let (S_n) be a (UM)-sequence of operators on E. If $E/\overline{\cup S_n E}$ is separable, or equivalently, if $\cap S_n'^{-1}\{0\}$ is separable, then $(1 - S_n)$ converges uniformly to a projection Q of finite rank with $\cap S_n^{-1}\{0\}$ as range and $\overline{\cup S_n E}$ as kernel.

Proof. Let $H = E/\overline{\cup S_n E}$ and $F_1 = \cap S_n'^{-1}\{0\}$. Then F_1 is (in a canonical way) the dual of H. Hence if F_1 is separable, H must be separable. Conversely, if H is separable, then as separable quotient of a Grothendieck space, H must be reflexive. Therefore, F_1 as the dual of a separable reflexive space must be separable.

Now we assume that H is separable. Then every bounded subset of F_1 is metrizable for the topology induced by $\sigma(E',E)$. Let $(x_n') \subset E'$ be bounded and let x' be a $\sigma(E',E)$-cluster point of the sequence $((S_n - 1)'x_n')$. It follows easily from (UM_2) that $S_m'x' = 0$ for all $m \in \mathbb{N}$. Hence $x' \in F_1$. Since $((S_n - 1)'x_n')$ is relatively $\sigma(E',E)$-compact, the set of all its $\sigma(E',E)$-cluster points is a bounded subset of F_1 and hence metrizable for the topology induced by $\sigma(E',E)$.

We conclude from Lemma 3 that the $\sigma(E',E)$-closure of $\{(S_n - 1)'x'_n\}$ is a compact metrizable space. Thus every sequence of this space has a $\sigma(E',E)$-convergent subsequence. Since E is a Grothendieck space, this subsequence converges also for $\sigma(E',E'')$. It follows from Eberlein's theorem that $((S_n - 1)'x'_n)$ is relatively weakly compact. Now, since (S'_n) is a (UM)-sequence, it follows from Lemma 4 that $(1 - S_n)''$ converges strongly to a weakly compact projection. This clearly implies that $1 - S_n$ converges also strongly to a weakly compact projection Q as well. Theorem 2 shows that $1 - S_n$ converges uniformly to Q and that $QE = \cap S_n^{-1}\{0\}$ and $(1 - Q)E = \overline{\cup S_n E}$. Finally, since E has the Dunford-Pettis property, we conclude that the weakly compact projection Q is of finite rank. Q.E.D.

We do not know whether the conclusion of Theorem 4 still holds if we only assume that $E/\overline{\cup S_n E}$ is reflexive.

4. APPLICATIONS

In this section we apply the results of Section 3 to Schauder decompositions, semigroups of operators, and ergodic theory.

An immediate consequence of Theorem 1 is:

Theorem 5 (Dean). Let E be a Grothendieck space with the Dunford-Pettis property. Then E does not have a (weak) Schauder decomposition.

Proof. Suppose that (P_n) is a (weak) Schauder decomposition of E and hence a (UM)-sequence with the operators P_n converging weakly to the identity. Now Theorem 1 implies that $\lim\|P_n - 1\| = 0$. Thus $P_n = 1$ for n sufficiently large, since the P_n are projections. But this contradicts the requirement for a (weak) Schauder decomposition that the P_n be distinct.

Theorem 1 was first proved in [8] using basic sequences. Our proof was originally given in [17], Corollaries 7 and 8.

Remark 3. It follows easily from Remark 1 that if (P_n) is a Schauder decomposition of a Banach space E with the Dunford-Pettis property, then (P''_n) is not a Schauder decomposition of E''.

We turn to semigroups of operators.

A family $\{T(t) : 0 \leq t\}$ (resp. $\{T(t) : 0 < t\}$) of bounded linear operators on a Banach space E is called a C_0-semigroup (resp. strongly continuous semigroup) if the following two conditions hold:

(i) $T(t + s) = T(t)T(s)$ for $0 \leq t,s$ (resp. $0 < t,s$)

(ii) $t \mapsto T(t)$ is a continuous map from $[0,\infty)$ (resp. $(0,\infty)$) into $\mathcal{L}(E)$ with the strong operator topology.

We note, that the C_0-semigroups are precisely the strongly continuous semigroups in the sense of [10], VIII.1.1. A C_0-semigroup $\{T(t) : 0 \leq t\}$ is called uniformly continuous if $t \mapsto T(t)$ is continuous for the uniform operator topology. If $\{T(t) : 0 < t\}$ is a strongly continuous semigroup such that $x = \lim_{t \to 0+} T(t)$ for every $x \in E$, then obviously, $\{T(t) : 0 \leq t\}$ with $T(0) = 1$ is a C_0-semigroup; in this case, we simply say that $\{T(t) : 0 < t\}$ is a C_0-semigroup.

In [14], Kishimoto and Robinson raise the question as to whether every C_0-semigroup of operators on L^∞ is uniformly continuous. This is answered affirmatively in [17], Theorem 3, where we give two proofs in the setting of a Grothendieck space with the Dunford-Pettis property (moreover, as pointed out in Remark 7 of [17], this also holds for semigroups of class (A) in the sense of [13]).

<u>Theorem 6</u>. Every C_0-semigroup of operators on a Grothendieck space with the Dunford-Pettis property is uniformly continuous.

<u>Proof</u>. Let $\{T(t) : 0 \leq t\}$ be a C_0-semigroup of operators on E. For every $n \in \mathbb{N}$, put $S_n = \int_0^1 T(t/n)dt$, where the integral is taken in the strong operator topology. We leave it to the reader to verify that (S_n) is a (UM)-sequence, that (S_n) tends strongly to the identity, and that

$$(*) \qquad \lim_{t \to 0+} \|T(t) - 1)S_m\| = 0 \quad , \qquad m \in \mathbb{N}$$

holds. Now, Theorem 1 implies that $\lim\|S_n - 1\| = 0$. Hence for some natural number, say m, $\|(S_m - 1)\| < 1$ and thus, S_m is invertible. It follows from $\|T(t) - 1\| \leq \|(T(t) - 1)S_m\|\|S_m^{-1}\|$ and (*) that $\lim_{t \to 0+} \|T(t) - 1\| = 0$. This in turn easily implies that $\{T(t) : 0 \leq t\}$ is uniformly continuous. Q.E.D.

As we have mentioned in the Introduction, Kishimoto and Robinson show in [14] that every C_0-semigroup on L^∞ is uniformly continuous provided that all operators are positive. Another special case of Theorem 6 is due to Rubel. His result, which can be found in [3], shows that every strongly continuous one-parameter group of isometries $\{T(t) : t \in \mathbb{R}\}$ on $H^\infty(D)$ is of the form $\{e^{\alpha it}1 : t \in \mathbb{R}\}$ for some $\alpha \in \mathbb{R}$ and hence, trivially, a uniformly continuous group.

Remark 4. Let $\{T(t) : 0 \leq t\}$ be a family of bounded linear operators on a Banach space E with the Dunford-Pettis property. If $\{T(t)'' : 0 \leq t\}$ is a C_0-semigroup of operators on E'', then $\{T(t) : 0 \leq t\}$ is a uniformly continuous C_0-semigroup (cf. [17], Theorem 4).

Theorem 7. Let $\{T(t) : 0 < t\}$ be a strongly continuous semigroup of operators on a Grothendieck space E with the Dunford-Pettis property and suppose that for some $a > 0$ the family $\{T(t) : 0 < t < a\}$ is uniformly bounded. If $E/\overline{\bigcup_{0<t} T(t)E}$ is separable, then there are bounded linear operators P and A on E with $1 - P$ a projection of finite rank and $PA = AP$ such that $T(t) = Pe^{At}$ for $t > 0$.

Proof. Let H be the closure of the linear subspace $\bigcup_{0<t} T(t)E$. By assumption, $\{T(t)\}$ is uniformly bounded on some non-empty interval $(0,a)$ and so on every interval $(0,b)$. This implies that the restrictions of $\{T(t) : 0 < t\}$ to H form a C_0-semigroup of operators on H and that $S_n = \int_0^1 T(t/n)dt$ exists in the strong operator topology. It is easily checked that (S_n) is a (UM)-sequence of operators on E. Let F and G be as in Lemma 1. Obviously, $G \subset H$. On the other hand, if $x \in H$, then $x = \lim_{t \to 0+} T(t)x$, and so, $x = \lim S_n x$. Hence $H \subset G$, and therefore $H = G$.

By assumption, E/G is separable. Now Theorem 4 implies that (S_n) converges to a projection P with $1 - P$ of finite rank, $PE = G$, and $(1 - P)E = F$. Hence G, as a complemented subspace of E, is a Grothendieck space with the Dunford-Pettis property. Theorem 6 implies that the restrictions of $\{T(t) : 0 < t\}$ to H form a uniformly continuous C_0-semigroup of operators on G. Hence there exists an operator $B \in \mathcal{L}(G)$ such that $T(t)_{|G} = e^{Bt}$ for $0 < t$ ([10], VIII.1.2). The operators T(t) and S_n commute. Hence P commutes with the operators T(t), which implies that $T(t)F \subset F$ for $0 < t$. But since $T(t)F \subset T(t)E \subset G$, the operators $T(t)$ vanish on F. Since E is the direct sum of F and G, $T(t) = Pe^{At}$, $0 < t$, for some $A \in \mathcal{L}(E)$ with $PA = AP$. Q.E.D.

We come to ergodic theory.

Let T be a bounded linear operator on a Banach space E. We denote the means $(\sum_{i=0}^{n-1} T^i)/n$ by T_n. The strong ergodic theorem ([10], VIII.5.1) asserts that if T^n/n tends strongly to zero and $(T_n x)$ is relatively weakly compact for every $x \in E$, then the means T_n converge strongly to a projection P with $PE = \{x: Tx = x\}$ and $(1 - P)E = \overline{(1 - T)E}$. The next theorem shows that on a Grothendieck space E with the Dunford-Pettis property the means T_n converge in the norm topology if the assumption that T^n/n tends strongly to zero is replaced by $\lim\|T^n\|/n = 0$. Theorems 8 and 9 were proved differently in [17].

Theorem 8 (Uniform Ergodic Theorem). Let T be a bounded linear operator on a Grothendieck space E with the Dunford-Pettis property. Suppose that $\|T^n\|/n$ tends to zero and that the means T_n are uniformly bounded. If for every $x \in E$ the sequence $(T_n x)$ has a weak cluster point, then the means T_n converge in the uniform operator topology.

Proof. Let $S_n = 1 - T_n$. We show that (S_n) is a (UM)-sequence. Clearly, if the means T_n are uniformly bounded, (UM_0) holds. It is easily checked that $1 - T_m = g_m(T)(1 - T)$ for some polynomial g_m. Hence if $\|T^n\|/n$ tends to zero, $\lim_n\|(1 - T_m)T_n\| = \lim_n\|g_m(T)(1 - T)T_n\| = \lim_n\|g_m(T)(1 - T^n)\|/n = 0$. This shows that (S_n) satisfies (UM_1) and (UM_2).

If for every $x \in E$ the sequence $(T_n x)$ has a weak cluster point, then $(S_n x)$ has a weak cluster point. Now Theorem 2 implies that (S_n) converges in the uniform operator topology. Hence the means $T_n = 1 - S_n$ converge in the uniform operator topology. Q.E.D.

Remark 5. Let T be a bounded linear operator on a Banach space E with the Dunford-Pettis property and suppose that $\|T^n\|/n$ tends to zero. If the means T''_n of T'' converge strongly, and the norm closure of $(1 - T'')$ is $\sigma(E'',E')$-closed, then the means T_n of T converge in the uniform operator topology. This follows from Remark 2. The condition on the norm closure of $(1 - T'')$ cannot be dropped: Let $E = \ell^1$ and let T be the operator $(\alpha_i) \mapsto ((1 - i^{-1})\alpha_i)$. Then (T''_n) converges strongly, but (T_n) does not converge in the uniform operator topology.

Theorem 9. Let T be a bounded linear operator on a Grothendieck space E with the Dunford-Pettis property. Suppose that $\|T^n\|/n$ tends to zero and that the means (T_n) are uniformly bounded. If $\{x' : T'x' = x'\}$ is separable, in particular, if it is finite dimensional, then the means T_n of T converge in the uniform operator topology to a projection of finite rank.

Proof. Let $S_n = 1 - T_n$. As we have seen in the proof of Theorem 8, the assumptions on T imply that (S_n) is a (UM)-sequence. Obviously, $\{x' : T'x' = x'\}$ is the kernel of S'_2. If this kernel is separable, then $\cap S_n'^{-1}\{0\}$ is separable. Now Theorem 4 implies that the operators $T_n = 1 - S_n$ converge in the uniform operator topology to a projection of finite rank. Q.E.D.

Special cases of Theorem 9 can be found in [15] and [16]. Finally, we note that Theorem 9 as well as the Scholium in [17] can be deduced from Theorems 2 and 4 above.

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POSITIVE BILINEAR FORMS AND THE RADON-NIKODYM THEOREM

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1. Introduction.

As is well known, most classical Banach spaces are Banach lattices. Compared to the enormous generality of (even separable) Banach spaces, the order structure constitutes a considerable enrichment in both geometric (see [7]) and operator theoretic (see [12]) terms. (For the elementary results of Banach lattice theory needed in the sequel, the reader is referred to either of those monographs.) If E,F denote (real) Banach lattices, a (bounded, linear) operator $T : E \to F$ is called *positive* ($T \geq 0$) if $Tx \geq 0$ for all $x \geq 0$ in E; T is called *order bounded* (or *regular*) if T is the difference of positive operators. The space of order bounded operators $E \to F$ is a Banach space $L^r(E,F)$ under the "regular" norm

$$\|T\|_r := \inf \|T_1 + T_2\|,$$

the infimum being taken over all decompositions $T = T_1 - T_2$ where $T_1 \geq 0$, $T_2 \geq 0$. If F is order complete (Dedekind complete), then for each order bounded T the modulus $|T| = \sup(T, -T)$ exists, $L^r(E,F)$ is a Banach lattice, and $\|T\|_r$ is simply the operator norm of $|T|$.

However, meaningful relations between order properties and topological properties of operators $T \in L^r(E,F)$ often are not easy to find. For example, there exist compact operators in ℓ^2 which are not order bounded, or which have a non-compact modulus $|T|$ (see [12, p. 231]). Or, if $T \geq 0$ has a certain topological property (such as compactness) and we have $0 \leq S \leq T$, what properties does S inherit from T? We will return to this question, recently settled by [1], below in some detail.

Combining expertise in Grothendieck's theory of tensor products [5] and familiarity with order bounded operators, Schlotterbeck [13] was probably the first to recognize that the point of closest contact between classical and order theoretic operator theory is the space of bounded linear operators $C(X) \to M(Z)$ (continuous real functions and Radon measures on the compact Hausdorff spaces X,Z, respectively). Considering, more generally, operators $T : E \to F'$ (E,F Banach spaces), Grothendieck [5] defined T to be *integral* if the bilinear form $(f,g) \to \langle Tf,g \rangle$ induces a continuous linear form on the ε-tensor product $E \otimes_\varepsilon F$. (For a brief introduction to integral maps, see [12, IV, 5]). The prototype of an integral map is the canonical injection $L^\infty(\mu) \to L^1(\mu)$ (μ finite), and integral maps $E \to F'$ are indeed characterized by admitting a factoring $E \to L^\infty(\mu) \xrightarrow[(\geq 0)]{} L^1(\mu) \to F'$ for a suitable Radon measure μ on some compact space. These maps form a Banach space $L^i(E,F')$ under the norm dual to the ε-norm on $E \otimes F$. Now if $E = C(X)$, $F = C(Z)$ (X,Z compact), the important fact is that the subspaces (of $L(C(X), M(Z))$) of integral and order bounded operators are identical, inclusive of their respective norms [12, IV. 5.6]:

<u>Theorem A.</u> *A linear map* $T : C(X) \to M(Z)$ *is integral iff it is order bounded, and* $\| T \|_i = \| T \|_r$.

2. The Operator Space $L^r(C(X), M(Z))$.

There is another way to look at order bounded operators $C(X) \to M(Z)$. By Thm. A a linear operator $T : C(X) \to M(Z)$ is order bounded iff the bilinear form $(f,g) \to \langle Tf,g \rangle$ on $C(X) \times C(Z)$ induces a continuous linear form φ_T on $C(X) \otimes_\varepsilon C(Z)$. But the completion $C(X) \tilde{\otimes}_\varepsilon C(Z)$ is isometrically isomorphic with $C(X \times Z)$ and hence, if T is order bounded, the continuous extension $\bar{\varphi}_T$ defines a Radon measure m_T on $X \times Z$. Conversely, every Radon measure m on $X \times Z$ defines, by virtue of $\varphi(f,g) := \int (f \otimes g)dm$, a continuous bilinear form φ on $C(X) \times C(Z)$ such that $\varphi(f,g) = \langle Tf,g \rangle$ for a uniquely defined operator $T : C(X) \to C(Z)' = M(Z)$, which is clearly order bounded. Moreover, $T \geq 0$ iff $m_T \geq 0$ and it is easy to see that the usual norm $\|m_T\|$ (total variation) equals the regular norm $\| T \|_r$.

We summarize:

<u>Theorem B.</u> *The mapping* $T \to m_T$, *given by the identity of bilinear forms*

$$\langle Tf,g \rangle = \int_{X \times Z} (f \otimes g)dm_T$$

on $C(X) \times C(Z)$, *is an isometric isomorphism of Banach lattices* $L^r(C(X),M(Z)) \to M(X \times Z)$ *(Radon measures on* $X \times Z$*). In particular, the principal band* $B(T)$ *of* $L^r(C(X), M(Z))$ *generated by* T *is isometrically isomorphic to the principal band of* $M(X \times Z)$ *generated by* m_T, *that is, to* $L^1(m_T, X \times Z)$.

The last assertion: $B(m_T) \cong L^1(m_T, X \times Z)$ is, of course, the classical Radon-Nikodym theorem applied to the measure m_T on $X \times Z$.

3. Approximation of Operators.

In his recent dissertation [6], W. Haid makes skilful use of Thm. B for the approximation of order bounded operators $T : E \to F'$ where $E = C(X)$, $F = C(Z)$ and, in fact, of operators between general Banach lattices. He thus obtains new proofs and extensions of results previously obtained by several authors (for details, see below). The present paper is intended to set the power and elegance of his approach in evidence; for details and technicalities, we must refer to [6] and a forthcoming paper by W. Haid. Haid's method of approximation can be outlined as follows. Let $T \in \mathcal{L}^r(C(X), M(Z))$ where, as above, X and Z are compact spaces, and let $S \in B(T)$ (the band generated by T). Then m_S (cf. Thm. B) is absolutely continuous with respect to $|m_T| = m_{|T|}$; therefore $m_S = h.\, m_T$ for a suitable $h \in L^1(m_{|T|})$ (Radon-Nikodym). Now $C(X \times Z)$ and, consequently, $C(X) \otimes C(Z)$ is dense in $L^1(m_{|T|})$; so h can be approximated, in the L^1-norm, by continuous functions on $X \times Z$ which are of the form $\sum_{i=1}^{n} p_i \otimes q_i$ ($p_i \in C(X)$, $q_i \in C(Z)$) (*Separation of Variables*).

The operator R corresponding, by Thm. B, to the measure $(\sum_{i=1}^{n} p_i \otimes q_i).\, m_T$ is given by the bilinear form

$$\langle Rf,g \rangle = \int_{X \times Z} (f \otimes g)\, (\sum_1^n p_i \otimes q_i) dm_T = \sum_1^n \int (p_i f \otimes q_i g) dm_T$$

$$= \sum_1^n \langle T(p_i f),\, q_i g \rangle = \langle (\sum_1^n \bar{q}_i T \bar{p}_i) f, g \rangle$$

on $C(X) \times C(Z)$, where by $\bar{p}_i$ and $\bar{q}_i$ we have denoted the operators (orthomorphisms) on $C(X)$ and $M(Z)$, respectively, defined by multiplication with the continuous functions p_i and q_i.

From the preceding identity we conclude:

Theorem C. *Let* $T \in \mathcal{L}^r(C(X), M(Z))$. *Then the set of all operators*

$$(*) \qquad \sum_{i=1}^{n} \bar{q}_i T \bar{p}_i ,$$

where $n \in \mathbb{N}$ *and* $\bar{q}_i$, $\bar{p}_i$ *are the orthomorphisms of* $M(Z)$, $C(X)$ *defined by arbitrary functions* $p_i \in C(X)$, $q_i \in C(Z)$ $(i = 1,\dots,n)$, *is dense in the principal band* $B(T)$ *of* $L^r(C(X), M(Z))$.

Corollary. *Let* H *be a closed linear subspace of* $L^r(C(X), M(Z))$ *such that* $T \in H$ *implies* $\bar{q}T\bar{p} \in H$ *for all* $p \in C(X)$, $q \in C(Z)$. *Then* H *is a band.*

The following are instructive examples.

1. The nuclear operators $C(X) \to M(Z)$ form a band in $L^r(C(X), M(Z))$ (cf. [12, IV. 5.6 and IV. 9.1]).

2. The order bounded compact operators $C(X) \to M(Z)$ form a band in $L^r(C(X),M(Z))$.

The second of these results was proved by Dodds and Fremlin [3] in 1978; in 1975 Fremlin [4] had shown that a positive compact operator $C(X) \to M(Z)$ is not necessarily nuclear. If X and Z are Stonian spaces, the operators in (*) can obviously be chosen to have the form $\Sigma_i \alpha_i \bar{q}_i T \bar{p}_i$ with p_i, q_i characteristic functions of open-and-closed subsets. In a forthcoming paper and in a more algebraic setting, B. de Pagter [11] uses such components $\bar{q} T \bar{p}$ of T $(T \geq 0)$ as well as more general ones for varied purposes of approximation (see also [2]).

4. Riesz Homomorphisms.

Let $T : C(X) \to M(Z)$ be order bounded; it is easy to see that the range of every $S \in I(T)$ (the ideal generated by T) and, in fact, of every $S \in B(T)$ is contained in the principal band of $M(Z)$ generated by $\mu := |T|e_X$ (e_X the unit of $C(X)$), that is, in $L^1(\mu,Z)$. It will be seen that $I(T)$ is especially easy to characterize if T or its adjoint is a Riesz homomorphism. Without restriction of generality we can assume that $L^1(\mu)$ is represented over the Stone space of the measure algebra of μ, so that Z is compact Stonian, μ a normal measure on Z and $L^\infty(\mu)$ can be identified with $C(Z)$.

Suppose now $T : C(X) \to L^1(\mu,Z)$ is a Riesz homomorphism (i.e. $|Tf| = T|f|$ for all $f \in C(X)$) and, to simplify, that $Te_X = e_Z$. It is well known [12, III. 9.1] that $Tf = f \circ \psi$ for some continuous $\psi : Z \to X$. Then for arbitrary $f \in C(X)$, $g \in C(Z)$ we obtain

$$< Tf,g > = < f \circ \psi,g > = \int_Z (f \circ \psi)g \, d\mu = \int_{G_\psi} (f \otimes g)dr,$$

where $r := \mu \circ \psi^{-1}$ is a Radon measure on $X \times Z$ supported by the graph G_ψ of ψ. From Thm. B we conclude that $m_T = r$ is supported by G_ψ. Conversely, if $T : C(X) \to M(Z)$ is positive, $Te_X = e_Z$, and m_T is supported by G_ψ for some continuous $\psi : Z \to X$, then T is a Riesz homomorphism (namely, $Tf = f \circ \psi$). It is a mere technicality to remove the assumptions $Te_X = e_Z$ and $T \geqslant 0$; hence, the operators S in the ideal $I(T)$ generated by a Riesz homomorphism T are characterized by the fact that m_S (see Thm. B) has its support contained in the graph of a continuous map $Z \to X$. (These operators S have $|S|$ a Riesz homomorphism.)

A positive operator T between Banach lattices E,F is called *interval preserving* (or to have the *Maharam property* [8]) if for all $0 \leqslant x \in E$, $T[0,x] = [0,Tx]$. This property is dual to the property of being a Riesz homomorphism in the sense that $T : E \to F$ is a Riesz homomorphism iff $T' : F' \to E'$ is interval preserving. Using this duality, Haid [6] shows that an order continuous operator $T : C(X) \to L^1(\mu,Z)$ is interval preserving iff there exists a continuous map $\varphi : X \to Z$ such that m_T has its support contained in $G_\varphi \subset X \times Z$. This leads to the following result:

<u>Theorem D.</u>

(i) *If* $T : C(X) \to L^1(\mu,Z)$ *is a Riesz homomorphism, then the ideal* $I(T)$ *consists of all operators* $\bar{q}T$, $\bar{q}$ *ranging over the set of all orthomorphisms of* $L^\infty(\mu,Z)$.

(ii) *If* $C(X)$ *is order complete,* $T : C(X) \to L^1(\mu,Z)$ *is order continuous and interval preserving, then the ideal* $I(T)$ *consists of all operators* $T\bar{p}$, $\bar{p}$ *ranging over the set of all orthomorphisms of* $C(X)$.

As an indication of proof for (i), let $Tf = f \circ \psi$ $(f \in C(X))$ and let $0 \leqslant S \leqslant T$; we recall that $L^\infty(\mu,Z)$ is identified with $C(Z)$. Then m_S has its support in G_ψ by the above; letting $\rho : G_\psi \to Z$ be defined by $(\psi(z),z) \to z$, $\omega := m_S \circ \rho^{-1}$ is a Radon measure on Z for which

$$< Sf,g > = \int_{G_\psi} (f \otimes g)\, dm_S = \int_Z (f \circ \psi) g\, d\omega \quad (f \in C(X),\ g \in C(Z)).$$

Letting $q := Se_X$ we obtain $< Se_X,g > = \int g \, d\omega = \int g\, q\, d\mu$ for all $g \in C(Z)$. Thus $\omega = q \cdot \mu$ und $Sf = q(f \circ \psi)$ for all $f \in C(X)$, whence $S = \bar{q}\, T$. Assertion (ii) follows similarly, using the duality of the Riesz and Maharam properties mentioned above.

Theorem D is a special (but typical) case of a much more general relationship (Thm. D' below).

5. Operators between Banach Lattices.

We will now consider the extension and application of the above ideas to operators between Banach lattices. Our discussion will be simplified by the following hypotheses on the Banach lattices E,F; this assumption can often (if not always) be removed.

(H) E is a Banach lattice possessing a dense principal ideal E_x ([12, II. 6]); F is an order complete Banach lattice on which there exists a strictly positive, order continuous linear form y'.

We recall that the ideal E_x generated by an element $x \in E_+$ is identical to $\bigcup_{n=1}^{\infty} nB$, where $B = \{z \in E: -x \leq z \leq x\}$ is a bounded, absolutely convex and complete subset of E; hence E_x is a Banach lattice (with unit ball B and order unit x). Dually, $y \to \langle |y|,y' \rangle$ defines a lattice norm on F which is additive on F_+; the completion (F,y') of F is a Banach lattice. By the well known theorems of Kakutani and Krein (cf.[12, II. 7,8]), we have $E_x \cong C(X)$ and $(F,y') \cong L^1(\mu,Z)$ (isometric isomorphisms of Banach lattices) for suitable compact spaces X,Z. Moreover, the canonical injections $i : C(X) \cong E_x \to E$ and $j : F \to (F,y') \cong L^1(\mu,Z)$ each define a Riesz isomorphism with range a dense ideal.

In these circumstances, it is easy to verify that bicomposition $T \to j \circ T \circ i$ is a Riesz isomorphism Φ of $\mathcal{L}^r(E,F)$ onto an ideal of $\mathcal{L}^r(C(X), L^1(\mu))$ which is dense in the strong operator topology. It is now a standard procedure to translate Thm. D into the following result.

THEOREM D'. *Let* E,F *be Banach lattices satisfying* (H).

(i) *If* $T : E \to F$ *is a Riesz homomorphism, then* $I(T)$ *is the set of all operators* $\bar{q}\, T$, *where* $\bar{q}$ *ranges over the orthomorphisms of* F.

(ii) *If* E *is order complete,* $T : E \to F$ *order continuous and interval preserving, then* $I(T)$ *is the set of all operators* $T\,\bar{p}$, *where* $\bar{p}$ *ranges over the orthomorphisms of* E.

For Banach lattices satisfying (H), this is the Radon-Nikodym type theorem for operators proved by Luxemburg and Schep [8] in a more algebraic setting.

Similarly, the approximation theorem (Thm. C above) can be adapted (via the map Φ) to the present situation, as follows.

THEOREM C'. *Let* E,F *be Banach lattices satisfying* (H). *If* $T : E \to F$ *is order bounded, then the set of operators*

$$(*) \qquad \sum_{i=1}^{n} \bar{q}_i \, T \, \bar{p}_i \, ,$$

where $n \in \mathbb{N}$ *and* $\bar{p}_i, \bar{q}_i$ *range over* Orth (E) *and* Orth (F), *respectively, is super order dense in the ideal* I(T) *of* $L^r(E,F)$ *(i.e., every* $S \in I(T)$ *is the order limit of a sequence of operators of this type).*

Comparing this to Thm. C above, the reader will notice that topological approximation by operators (*) is replaced by approximation in order; this is of course, due to the fact that the mapping Φ (see above) is continuous but not, in general, a homeomorphism. In fact, if $T : E \to F$ is compact and $0 \leq S \leq T$ then (in contrast with the situation in $L(C(X), M(Z))$; see Sect. 3, Example 2) in general S is no longer compact [1]. Therefore, in order to obtain topological approximation through (*), one has to search for conditions (on E,F and/or T) under which sequential order convergence in I(T) implies convergence in norm. Here is one example. If E' and F have order continuous norm and $T \geq 0$ is compact, then by modifying an argument of Nagel and Schlotterbeck [10], Haid [6] shows the norm of $L^r(E,F)$ to be order continuous on I(T). It follows immediately from Thm. C' that in these circumstances, every operator in I(T) is compact - a result first proved by Dodds and Fremlin [3] in 1979.

If we call an operator $S : E \to F$ satisfying $0 \leq S \leq T$ for some compact T, *compactly majorized*, then a natural question is: What properties are enjoyed by compactly majorized operators in general?

After the partial answer by Dodds and Fremlin [3] just mentioned, Aliprantis and Burkinshaw [1] solved the problem of compactness of S as follows: The composition of at least three compactly majorized operators is always compact, and no fewer than three will do in general.

On the basis of Thm. C', Haid [6] obtains several generalizations and refinements of the theorem of Aliprantis and Burkinshaw. (Similar results have simultaneously been obtained by B. de Pagter [11] with different methods.) Of Haid's results we only mention the following.

Theorem E. *Let* G,E,F,H *denote Banach lattices and let* $T_1 : G \to E$, $T_2 : E \to F$, *and* $T_3 : F \to H$ *denote positive operators such that* T_1, T_3 *are weakly compact* *and* T_2 *is compact.*

If S_i *are operators satisfying* $0 < S_i < T_i$ $(i = 1,2,3)$, *then* $S_3 \circ S_2 \circ S_1 : G \to H$ *is compact.*

The proof of this theorem, which is also independent of (H), is lengthy and technical. Thus for this and similar results, especially on so-called Dunford-Pettis operators (i. e., operators transforming weakly convergent into norm convergent sequences), the interested reader is referred to [6].

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Functional Analysis: Surveys and Recent Results III
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WHAT CAN POSITIVITY DO FOR STABILITY ?

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For strongly continuous semigroups $(T(t))_{t\geq 0}$ of positive linear operators on Banach lattices it is shown that the asymptotic behavior, i. e. $\lim_{t\to\infty} T(t)$, depends strongly on the location of the spectrum $\sigma(A)$ of the generator A .

1. Introduction

It is one of the basic results on linear differential equations that the location of the eigenvalues of the matrix $A = (a_{ij})_{n\times n}$ determines the asymptotic behavior of the solutions of the Cauchy problem

$$(*) \qquad \frac{d}{dt} x(t) = Ax(t) \quad , \qquad x(0) = x_0 \in \mathbb{C}^n .$$

More precisely, if

(**) $\quad$ $\mathrm{Re}\,\lambda \leq \varepsilon$ for all elements λ of the spectrum $\sigma(A)$ and some $\varepsilon < 0$,

then all solutions $x(t) := e^{tA} x_0$ of (*) are <u>stable</u> , i. e.

$$(\overset{*}{*}) \qquad \lim_{t\to\infty} \| e^{tA} x_0 \| = 0 \quad \text{for every } x_0 \in \mathbb{C}^n \text{ , resp.}$$

$$(\substack{**\\**}) \qquad \lim_{t\to\infty} \| e^{tA} \| = 0 \quad ,$$

since strong and uniform convergence coincide on finite dimensional Banach spaces.

It is clearly of great importance to look for an infinite dimensional version of the above result. Therefore we replace $\mathbb{C}^n$ by an arbitrary Banach space E and the matrix (a_{ij}) by a linear operator A with dense domain $D(A)$ in E . Then the 'abstract' Cauchy problem (*) is well posed for every initial value $x_0 \in D(A)$

if and only if A is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. In that case the solutions of (*) are given by $x(t) := T(t)x_0$. We refer to [5], in particular to theorem 2.3.2 for more details.

Again we may ask whether the location of the spectrum $\sigma(A)$ determines the 'stability' of the semigroup $(T(t))_{t\geq 0}$, i.e. of the solutions of (*), but it is evident that in the infinite dimensional context there are different concepts of 'stability' naturally generalizing $({}^{**}_{*})$, resp. $({}^{**}_{**})$.

2. Uniform Stability

Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup of bounded, linear operators with generator $(A,D(A))$ on some Banach space E (see [3] for the basic definitions). Then the location of the spectrum $\sigma(A)$ of A and the asymptotic behavior of $\mathcal{T}$ with respect to the norm topology on $\mathcal{L}(E)$ can be described by the following two constants.

2.1 Definition: The spectral bound of A is

$$s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}.$$

The growth bound of $\mathcal{T}$ is

$$\omega := \omega(\mathcal{T}) = \inf\{w \in \mathbb{R} : \exists\ M_w \text{ such that } \|T(t)\| \leq M_w \cdot e^{wt},\ t\geq 0\}.$$

From general semigroup theory it follows that always

$$s(A) \leq \omega(\mathcal{T}).$$

Moreover one knows that $\lim_{t\to\infty} \|T(t)\| = 0$, i.e. $\mathcal{T}$ is uniformly stable, if and only if $\omega < 0$. Therefore uniform stability is characterized by the location of $\sigma(A)$ if we can answer positively the following:

2.2 Question: $s(A) = \omega(\mathcal{T})$?

There are a number of examples showing that in general we may have $s(A) < \omega$ (see [3], 2.17, [10], [17]), but the following one (due to M. Wolff) is particularly simple.

2.3 Example: Take the translation semigroup

$$T(t)\, f(x) := f(x+t)$$

on $\quad E := C_0(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, e^x dx)$

with norm $\|f\| := \|f\|_\infty + \|f\|_1$. Then $s(A) \square -1$, while $\omega \square 0$. Remark that spectral and growth bound coincide for the translation semigroup on each of the two spaces C_0 , resp. L^1 . In fact, they are zero on $C_0(\mathbb{R}_+)$ and -1 on $L^1(\mathbb{R}_+, e^x dx)$.

The spectral radius of the operators $T(t)$, $t > 0$, is determined by the growth bound by the formula

$$r(T(t)) = e^{\omega t} \quad , \quad ([3]\ , 1.22) \quad .$$

Therefore the spectral mapping theorem for norm continuous or merely eventually norm continuous semigroups ([3], 2.19) implies $s(A) = \omega$. Except for self adjoint semigroups on Hilbert spaces no other general answer to question (2.2) seemed to be known. But since positive operators on ordered Banach spaces admit a rich spectral theory - today called Perron-Frobenius theory - , it was tempting to ask the question formulated in the title of this talk: For which semigroups of positive operators on which ordered Banach spaces does $s(A) \square \omega$ hold ?

The semigroup in (2.3) indicates that we should not be too optimistic. In fact, in this example E is a Banach lattice (see [16]) and each $T(t)$ is a positive operator. Therefore, in order to obtain '$s(A) \square \omega$' we need some additional hypothesis either on the semigroup $(T(t))$ or on the Banach space E .

It was R. Derndinger [4] who first realized that it is a simple consequence of the Krein-Rutman theorem that '$s(A) \square \omega$' holds for every positive semigroup on spaces $C(X)$, X compact. The subsequent investigation of other classical Banach lattices brought up a number of interesting results.

2.4 Theorem (Derndinger [4], Greiner-Nagel [9]) : Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup of positive operators on $E = L^p(X,\mu)$ for $p = 1,2,\infty$. Then $s(A) \square \omega$.

Proof: For a semigroup of positive operators the integral representation of the resolvent $R(\lambda,A) := (\lambda-A)^{-1}$ holds not only for $\mathrm{Re}\,\lambda > \omega(\mathcal{T})$ (see [3], 2.8) but already for $\mathrm{Re}\,\lambda > s(A)$. In fact, the following is true:

For each $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\,\lambda > s(A)$ the resolvent $R(\lambda,A)$ of the generator A is given as

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda s}\, T(s)f\, ds \quad , \quad f \in E \quad ,$$

and therefore $\| R(\lambda,A)\| \le \| R(\mathrm{Re}\,\lambda, A)\|$.

For a proof we refer to [4], 3.2 or [10], 3.3 .

Case p = 1 (G. Greiner): By considering a 'rescaled' semigroup $(e^{\alpha t} T(t))_{t\geq 0}$ for appropriate $\alpha \in \mathbb{R}$ we may assume $s(A) < 0$. Then, for $\lambda \geq 0$, $\lambda R(\lambda,A)$ is uniformly bounded and, as remarked above,

$$R(\lambda,A) = \int_0^\infty e^{-\lambda s}\, T(s)\, ds \quad .$$

Therefore we have for $0 \le f \in E$ and $0 < t$

$$\frac{1}{t}\int_0^t T(s)f\, ds \le \frac{e}{t}\int_0^t e^{-s/t}\, T(s)f\, ds \le \frac{e}{t}\int_0^\infty e^{-s/t}\, T(s)f\, ds = \frac{e}{t} R(\tfrac{1}{t},A)f \ ,$$

and hence the Cesaro means

$$C(t) := \frac{1}{t}\int_0^t T(s)\, ds$$

are uniformly bounded for $t > 0$.

Next choose constants $w > 0$, $M \geq 1$ such that $\| T(t)\| \le M\cdot e^{wt}$ for $t \geq 0$.

For $0 \le s \le t$ and $0 \le f \in E$ we obtain

$$\| T(t)f\| \le \| T(t-s)\| \; \| T(s)f\| \le M\cdot e^{w(t-s)} \; \| T(s)f\|$$

or $$M^{-1}\, e^{w(s-t)}\, \| T(t)f\| \le \| T(s)f\| \ .$$

Integration from 0 to t and the additivity of the L^1-norm on positive functions yield

$$\| T(t)f\| \le \frac{w\,Mt}{1-e^{-wt}}\; \frac{1}{t}\int_0^t \| T(s)f\|\, ds = \frac{w\,Mt}{1-e^{-wt}}\; \| C(t)f\| \le M_1(t+1)\| f\|$$

for some constant $M_1 \geq 1$ and all $t \geq 0$. Therefore

$$\| T(t) \| \leq M_1(t+1) \quad ,$$

i. e. the growth of $\| T(t) \|$ is of polynomial order. From the definition of the growth bound we conclude $\omega \leq 0$.

If $s(A)$ is strictly smaller than ω we apply the above considerations to the rescaled semigroup $(e^{-vt}T(t))_{t\geq 0}$ for $v := \frac{1}{2}(s(A) + \omega)$ and obtain a contradiction. Therefore

$$s(A) = \omega$$

holds.

Case p = 2 (see [9]): The proof is a combination of two important results. The first is the fact that for positive semigroups the norm of the resolvent $R(\lambda,A)$ can be estimated by the norm of the resolvent in $\mathrm{Re}\,\lambda$ (see the beginning of this proof). The second is true for Hilbert spaces only and essentially due to Gearhart [6] :

> The growth bound of a strongly continuous semigroup $(T(t))$ on a Hilbert space H is characterized by
>
> $\omega = \inf\{\alpha \in \mathbb{R} : \alpha + i\mathbb{R} \in \varrho(A)$ and $\{R(\alpha+i\beta) : \beta \in \mathbb{R}\}$ is uniformly bounded$\}$.

Various proofs of this characterization can be found in [6], [9] or most recently [15] , but we point out that it does not hold in example (2.3).

Case $p = \infty$: In fact, this is a special case of the above mentioned result for positive semigroups on spaces $C(X)$ (see [4]), but more generally it follows from a beautiful result of H. P. Lotz on the automatic norm continuity of strongly continuous semigroups on L^∞-spaces (see [12] or Lotz's article in this volume).

Remark: Further results on the coincidence of ω and $s(A)$ for positive semigroups on $C_0(X)$, X locally compact, and on C^*-algebras have been proved by Batty-Davies [2] and Groh-Neubrander [9] .

3. Strong Stability

As we have seen, the condition $s(A) < 0$ in general does not imply uniform stability of the semigroup, but there is still some hope to obtain a weaker stability property, e. g. one corresponding to $(^{*}_{*})$. Moreover, we point out that the actual solutions of the Cauchy problem $(*)$ are given by

$$t \mapsto T(t)x$$

for $x \in D(A)$ only. Therefore it will be good enough for many purposes to have some sort of stability for these solutions only. This idea was developped by F. Neubrander in [13] where he defines the following.

3.1 Definition: A semigroup $(T(t))_{t\geq 0}$ on a Banach space E is called strongly (exponentially) stable if there exists $v < 0$ such that $\|T(t)x\| \leq M_x \cdot e^{vt}$ for every $x \in D(A)$, $t \geq 0$ and appropriate constants M_x.

Since the following result does not hold for each strongly continuous semigroup on an arbitrary Banach space it shows again the usefulness of positivity for stability theory.

3.2 Theorem (Neubrander [13]): Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup of positive operators on a Banach lattice E. If the spectral bound of the generator is smaller than zero, i. e. $s(A) < 0$, then the semigroup is strongly stable.

4. Spectral Decompositions

In this section we look for a more detailed description of the asymptotic behavior of semigroups $(T(t))_{t\geq 0}$. As before we hope to obtain such a description from information on the spectrum $\sigma(A)$ of the generator A. More precisely, we ask the following:

4.1 Question: Given a strongly continuous semigroup $(T(t))_{t\geq 0}$ on some Banach space E. Assume that the spectrum $\sigma(A)$ of the generator decomposes in the following way:

$$\sigma(A) = \sigma_1 \cup \sigma_2 \quad ,$$

where $\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma_1\} < \alpha < \inf\{\mathrm{Re}\,\lambda : \lambda \in \sigma_2\}$

for some $\alpha \in \mathbb{R}$. Does there exist a 'spectral decomposition' of $(T(t))$ and E corresponding to the decomposition of $\sigma(A)$? This means: Can we find closed, $(T(t))$-invariant subspaces E_1, E_2 such that $E = E_1 \oplus E_2$ and

$\| T(t)x_1 \| \le M e^{\alpha t} \| x_1 \|$, $\| T(t)x_2 \| \ge m e^{\alpha t} \|x_2\|$ for all $x_1 \in E_1$, $x_2 \in E_2$, $t \ge 0$

and appropriate constants $m, M > 0$?

Such spectral decompositions do not follow from the usual spectral calculus since both σ_1 and σ_2 may be unbounded. Moreover, the example (2.3) shows - for $\sigma_2 = \emptyset$ - that in general the norm estimates are not satisfied. Nevertheless we present two interesting results on the existence of such spectral decompositions and in both cases the positivity of the semigroup is essential. Remark that the underlying Banach lattices are again those for which we could already answer (2.2).

4.2 Theorem (Arendt-Greiner [1]): Assume $(T(t))_{t \in \mathbb{R}}$ to be a strongly continuous group of positive operators on $C_0(X)$, X locally compact. Then spectral decompositions as explained above are possible.

4.3 Theorem: Let $\mathcal{T} = (T(t))_{t \ge 0}$ be a bounded, irreducible semigroup of positive operators on $E = L^2(X,\mu)$. Assume that $s(A) = 0$ is a pole of the resolvent of the generator A and that $\sigma(A) \cap i\mathbb{R} \neq \{0\}$. Then the following holds:

(i) $\sigma(A) = \sigma_1 \cup \sigma_2$, where $\sigma_1 = i\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$ and $\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma_2\} < 0$.

(ii) $E = E_1 \oplus E_2$, where E_1 is a closed sublattice isomorphic to $L^2(\Gamma, m)$, Γ the unit circle.

(iii) $(T(t)|_{E_1})_{t \ge 0}$ is a group isomorphic to a rotation group on $L^2(\Gamma, m)$.

(IV) The growth bound $\omega(T(t)|_{E_2})$ is smaller than 0 .

Proof: Since $\mathcal{T}$ is irreducible and 0 is assumed to be a pole of the resolvent we conclude from [7], 2.5.b that $\sigma(A) \cap i\mathbb{R}$ is a subgroup $i\alpha\mathbb{Z}$ of $i\mathbb{R}$ consisting only of simple poles of the resolvent. Moreover [8], 2.6.c implies $\sigma(A) = \sigma(A) + i\alpha\mathbb{Z}$ and $\| R(\lambda) \|$ ▫ $\| R(\lambda + in\alpha) \|$ for every $\lambda \in \varrho(A)$, $n \in \mathbb{Z}$.

This proves (i) and shows in addition that $\| R(\lambda) \|$ is bounded on each imaginary axis $-\gamma + i\mathbb{R}$ with sufficiently small $\gamma > 0$.

Next we apply the Glicksberg-de Leeuw theory to the abelian, relatively weakly compact semigroup $\mathcal{T}$ and obtain a decomposition of E into the closed subspace E_1 of all reversible vectors and the closed subspace E_2 of all escape vectors (e. g. see [16], p. 214). Then it is known that E_1 is spanned by all eigenvectors pertaining to purely imaginary eigenvalues of A. Therefore $(T(t)|_{E_1})_{t\geq 0}$ has discrete spectrum and its weak (=strong) closure yields a compact group.
The projection P onto E_1 with kernel E_2 is orthogonal and strictly positive ($\mathcal{T}$ is irreducible), hence [16], III. 11.5 & 11.6 imply that E_1 ▫ PE is a sublattice of E. On this sublattice the restricted semigroup $(T(t)|_{E_1})$ is still irreducible (having one-dimensional fixed space) and has discrete spectrum. Therefore the Halmos-v. Neumann theorem (e. g. [16], III. 10.4) assures: E_1 is isomorphic to the L^2-space on the dual group of $\sigma(A) \cap i\mathbb{R}$ ▫ $i\alpha\mathbb{Z}$ and $(T(t)|_{E_1})_{t\geq 0}$ is isomorphic to a rotation group on $\Gamma = \hat{\mathbb{Z}}$ (whose period is determined by the smallest non-zero eigenvalue $i\alpha$). Hence we proved (ii) and (iii).

Finally, observe that $(T(t)|_{E_2})$ is a strongly continuous semigroup on the Hilbert space E_2 such that $s(A|_{E_2}) < 0$ and the resolvent of $A|_{E_2}$ is bounded on imaginary axes $-\gamma + i\mathbb{R}$ for small $\gamma > 0$. From Gearhart's result mentioned in section 2 (see: proof for $p = 2$) it follows $\omega(T(t)|_{E_2}) < 0$, i.e. (iv).

Final remark: Further results on the asymptotic behavior of positive semigroups with respect to the strong operator topology can be found in [8].

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EXTENSION OF POSITIVE OPERATORS

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Let E, F denote Banach lattices, H a linear subspace of E and let $T:H\to F$ be a continuous positive operator. In order to solve problems concerning the existence and uniqueness of positive extensions $T_o: E\to F$ of T it is necessary to study sublinear operators from E into abstract cones containing F (see [2]). The purpose of this note is to prove some useful identities for sublinear operators occuring in this context. As a consequence an easy access to results on extensions of positive operators in classical Banach lattices is achieved.

INTRODUCTION

Let E be a locally convex vector lattice, F a Dedekind complete topological vector lattice and H a linear subspace of E. If $T:H\to F$ is a continuous positive operator we shall be concerned with the following two problems:

1. Do there exist positive extensions $T_o: E\to F$ of T?
2. Describe $\{T_o e : T_o$ positive extension of $T\}$ for each $e \in E$!

In a previous publication (see [2]) these two problems were satisfactorily solved at least for E an L^p-space, F an L^q-space with $1 \leq q \leq p \leq \infty$. However, even in the L^p context, many questions remained open. In order to obtain extension results beyond the L^p case, it is of utmost importance to develop a well-established theory of sublinear mappings into (abstract) cones. For the readers' convenience let us first recall some basic definitions and results (see [5],[2],[3]).

1. PRELIMINARIES AND NOTATIONS

All vector spaces occuring in this note will be *real*.
If x, y are arbitrary elements of some lattice X we shall use the notations $x\wedge y$, $x \vee y$ for the infimum and the supremum of x,y, respectively. Correspondingly, for a finite family $(x_i)_{i\in I}$ in X

$$\bigvee_{i\in I} x_i := \sup\{x_i : i\in I\}, \quad \bigwedge_{i\in I} x_i := \inf\{x_i : i\in I\}.$$

Let F denote a Dedekind complete vector lattice. Then there exists a Dedekind complete lattice ordered cone F_s (in the terminology of [3], pages 6,7) containing F with the following properties

S1) The supremum $\sup A$ of A exists for every subset A of F_s, in particular, $\infty := \sup F_s \in F_s$.

S2) The elements of F are precisely those members of F_s which are invertible with respect to addition.

S3) For all subsets $A,B \subset F_s$ the equivalence

$$\sup A = \sup B \Leftrightarrow \forall_{f\in F} \sup_{a\in A}(a\wedge f) = \sup_{b\in B}(b\wedge f)$$

is valid.

S4) $\{x \in F_s : x \leq f\} \subset F$ for each $f \in F$.

S5) $f + \sup A = \sup_{a\in A}(f + a)$ for every subset A of F and each $f\in F$.

F_s is called the *sup-completion* of F and is uniquely determined up to isomorphisms of ordered cones (for proofs see [2], p. 6 - 10). For instance, the sup-completion of $\mathbb{R}$ is $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$, the sup-completion of a space $L^p(\mu)$, $p \in [1,\infty[$, μ a σ-finite measure, is the cone of all L^p-minorized equivalence classes of $\mathbb{R}_\infty$-valued μ-measurable functions (the equivalence being μ-a. e. equality).
If F is a locally convex vector lattice, the sup-completion of the topological dual F' of F is the cone of all lower semicontinuous (abbreviated in the sequel by l.s.c.), additive, positively homogeneous functionals from F_+ into $\mathbb{R}_\infty$.

While in arbitrary lattice ordered cones G positive parts $x^+ := x\vee 0$ of elements $x\in G$ always exist, negative parts are not available, in general, since $x\wedge 0$ need not be additively invertible. If, however, G is the sup-completion of a (Dedekind complete) vector lattice, $x^- := -(x\wedge 0)$ is well-defined, since $x\wedge 0$ is invertible with respect to addition by (S4).
We adopt the usual notion of sublinearity to mappings between cones. In contrast to [3], however, a linear map will only attain values invertible under addition.

<u>1.1 Lemma:</u> Let F_s be the sup-completion of a Dedekind complete vector lattice F. Then $\sup\{f \in F : f \leq a\} = a$ for all $a \in F_s$.

<u>Proof:</u> Setting $A := \{f \in F : f \leq a\}$ we have $a \wedge g \in A$ for all $g \in F$ by (S4). Hence, for each $g \in F$,

$$\sup_{f\in A}(f \wedge g) \geq (a \wedge g) \wedge g = a \wedge g \geq \sup_{f\in A}(f \wedge g)$$

which yields $\sup A = \sup\{g\} = g$ by (S3). ∎

Let P denote a sublinear mapping from a locally convex space E into the sup-completion F_s of a Dedekind complete topological vector lattice F. If, for each $e \in E$, $K_{P,e}$ denotes the set of all concave continuous maps $k : U \to F$ defined on some convex neighborhood of e, then the *regularization* $P^{\cap}$ of P is defined by

$$P^{\cap}e = \sup\{k(e) : k \in K_{P,e}\}$$

$P^{\cap}$ is a sublinear operator dominated by P provided that $K_{P,0} \neq \emptyset$ and

$$P^{\cap}e = \sup\{Te : T : E \to F \text{ linear continuous, } T \leq P\}$$

for all $e \in E$ (for proofs see again [2]). P will be called *regularized* if $P^{\cap} = P$. Note that a sublinear *functional* $p : E \to \mathbb{R}_{\infty}$ is regularized iff it is l.s.c..

<u>1.2 Example:</u> Let X be a compact space, F a Dedekind complete topological vector lattice, $S : C(X) \to F$ a positive operator and H a linear subspace of $C(X)$ (not necessarily containing a strictly positive function). Given a function $g \in C(X)$, is it possible to find a positive modification T of S such that $S|_H = T|_H$ and $Sg \neq Tg$?

To answer this question, consider the mapping $P : C(X) \to F_s$ defined by

$$Pe = \sup_{\varepsilon>0} \inf\{Sh : h \in H,\ h \geq e - \varepsilon\cdot 1\},$$

where 1 denotes the constant function with value 1 and we use the convention $\inf \emptyset := \infty$. It is easy to show that P is sublinear and that a linear operator $T : C(X) \to F$ is P-dominated if and only if it is a positive extension of $S|_H$. We claim that P is regularized. In order to prove this let $e \in E$ be given. If, for every $\varepsilon > 0$, $U_{\varepsilon/2}$ denotes the closed ball of center e and radius $\frac{\varepsilon}{2}$, then the implication

$$\forall_{h\in H} \left(h \geq k - \tfrac{\varepsilon}{2}\cdot 1 \Rightarrow h \geq e - \varepsilon\cdot 1\right)$$

is valid for each $k \in U_{\varepsilon/2}$. Thus, if $\{h \in H : h \geq e - \varepsilon\cdot 1\} = \emptyset$ for some $\varepsilon > 0$, then the constant function $\tilde{f} : U_{\varepsilon/2} \to F$ of value $f \in F$ is majorized by P on $U_{\varepsilon/2}$, since $P(k) \geq \inf\{Sh : h \in H,\ h \geq k - \frac{\varepsilon}{2}\cdot 1\} = \inf \emptyset = \infty$ for each $k \in U_{\varepsilon/2}$. It follows that $Pe = \sup_{f\in F} \tilde{f}(e) = \infty$, which shows that P is regularized at e.

If $\{h \in H : h \geqq e - \varepsilon \cdot 1\} \neq \emptyset$ for every $\varepsilon > 0$, the constant function $c_\varepsilon : U_{\varepsilon/2} \to F$ given by $c_\varepsilon(k) = \inf\{Sh : h \in H, h \geqq e - \varepsilon \cdot 1\}$ satisfies the inequality

$$c_\varepsilon(k) \leqq \inf\{Sh : h \in H, \ h \geqq k - \frac{\varepsilon}{2} \cdot 1\} \leqq P(k)$$

for each $k \in U_{\varepsilon/2}$, hence c_ε is dominated by P on $U_{\varepsilon/2}$ and

$$\sup_{\varepsilon > 0} c_\varepsilon(e) = Pe.$$

Therefore, P is regularized at e.

Consequently, we have Tg = Sg for every positive extension $T : C(X) \to F$ of $S|_H$ if and only if $Se = Pe$ and $S(-e) = P(-e)$. Since always $Pe \geqq Se$ and $P(-e) \geqq S(-e)$, this is equivalent to $Pe \leqq Se$ and $P(-e) \leqq S(-e)$ or, which amounts to the same,

$$\inf\{Sh : h \in H, h \geqq e - \varepsilon \cdot 1\} \leqq Se \leqq \sup\{Sh : h \in H, \ h \leqq e + \varepsilon \cdot 1\}$$

for every $\varepsilon > 0$.

2. SUBLINEAR OPERATORS AND BISUBLINEAR FUNCTIONALS

Let E be a real vector space and F a locally convex vector lattice. Since the sup-completion F'_s of the topological dual F' of F is (ordered cone isomorphic to) the cone of all l.s.c., additive, positively homogeneous $\mathbb{R}_\infty$-valued functions on F_+, a sublinear operator $P : E \to F'_s$ induces a bisublinear functional $p : E \times F \to \mathbb{R}_\infty$ defined by

$$p(e,f) = \begin{cases} Pe(f), & \text{if } f \geqq 0 \\ 0 \quad , & \text{if } e = 0 \\ \infty \quad , & \text{else} \end{cases}$$

The functional p obviously satisfies the conditions

i) $\{f : p(e,f) < \infty\} \subset F_+$ for each $e \in E \setminus \{0\}$,

ii) $f \to p(e,f)$ is l.s.c. additive for each $e \in E$.

Conversely, if $p : E \times F \to \mathbb{R}_\infty$ is bisublinear such that the conditions (i), (ii) hold then the equality $Pe(f) = p(e,f)$ $(f \geqq 0)$ defines a sublinear operator P from E into F'_s (this is also true without condition (i)). Setting

$$p^{\otimes}(t) \ := \inf\{\sum_{i \in I} p(e_i,f_i) : (e_i,f_i)_{i \in I} \in \mathcal{F}_t\} \quad \text{for } t \in E \otimes F,$$

where $\mathcal{F}_t$ is the set of all finite families $(e_i,f_i)_{i \in I}$ in $E \times F$ such that $\sum_{i\in I} e_i \otimes f_i = t$, we obtain a sublinear, $\mathbb{R}_\infty$-valued functional $p^\otimes$ on the tensor product $E \otimes F$. For F a Dedekind complete Banach lattice, it is known that $p(e,f) = p^\otimes(e \otimes f)$ for all $(e,f) \in E \times F_+$ (see [2], Thm. 3.17). The equality remains true, however, for arbitrary locally convex vector lattices. For readers interested in the details we add a sketch of a generalized proof at the end of this note. Therefore, the correspondence $p \to p^\otimes$ is one-to-one, provided that only bisublinear functionals p on $E \times F$ with the properties (i) and (ii) are admitted.

Every linear operator $T:E\to F'$ induces a bilinear form b_T on $E\times F$, where $b_T(e,f) = Te(f)$. This, in turn, generates a unique linear form $b_T^\otimes$ on $E \otimes F$. Moreover, T is P-dominated iff b_T is majorized by p, or equivalently, if $b_T^\otimes$ is dominated by $p^\otimes$. Provided that E is a locally convex space, F a normed vector lattice and F' bears the dual norm, the mapping $T\to b_T$ is a bijection from the set of all continuous linear operators $T:E\to F'$ onto the set of all continuous bilinear forms on $E\times F$ endowed with the product topology. On the other hand a bilinear form b on $E\times F$ is continuous iff $b^\otimes$ is continuous with respect to the projective topology on $E\otimes F$.

Setting $\mathfrak{T}_P := \{T:E\to F' : T \text{ continuous, linear, } T\le P\}$ it follows from [2], Thm. 2.8 and Lemma 2.11 that $\{Te: T\in\mathfrak{T}_P\}$ is upward directed for each $e\in E$. Hence

$$\begin{aligned} P^\cap e(f) &= (\sup_{T\in\mathfrak{T}_P} Te)(f) = \sup\{Te(f) : T \in \mathfrak{T}_P\} = \sup\{b_T^\otimes(e \otimes f) : T \in \mathfrak{T}_P\} \\ &= \sup\{\ell(e \otimes f) : \ell \in (E \otimes F)' ,\ \ell \le p^\otimes\} \\ &= q(e \otimes f) , \end{aligned}$$

where q denotes the regularization of $p^\otimes$. Therefore, *P is regularized iff* $p^\otimes$ *is l.s.c. with respect to the projective topology* on $E \otimes F$. In this case, p is l.s.c. on $E \times F$ endowed with the product topology. Unfortunately, *the converse is not generally true* (see [2], for counterexamples).

Restricting our attention to the extension of *positive* operators from a locally convex vector lattice E into the dual F' of a normed vector lattice F we shall be concerned with *isotone* sublinear maps $P : E \to F'_s$. Then p satisfies the additional condition

iii) $e \to p(e,f)$ is isotone for each $f \in F_+$.

For bisublinear functionals of this type the lower semicontinuity of p implies that of $p^\otimes$ at least for the majority of the classical Banach lattices. In fact, this implication is true in each of the following cases:

1) E is an arbitrary normed vector lattice, F a Dedekind complete AM-space,

2) E is an L^p-space, F and L^q-space, where $\frac{1}{p} + \frac{1}{q} \leq 1$, $p, q \in]1,\infty[$,
3) E is an AM-space and F an arbitrary Banach lattice

(see [2]). Thus, there are serious reasons to investigate lower semi-continuous bisublinear functionals on $E \times F$ satisfying the conditions (i),(ii),(iii).

In particular, the following problems have to be solved:

a) Given an isotone sublinear operator $P : E \to F'_s$, is the greatest l.s.c. minorant $\hat{p} : E \times F \to \mathbb{R} \cup \{-\infty,\infty\}$ of p defined by

$$\liminf_{(e,f)\to(e_o,f_o)} p(e,f) = \hat{p}(e,f) \quad \text{for all } (e_o,f_o) \in E \times F$$

again a bisublinear functional with range contained in $\mathbb{R}_\infty$? (If $\hat{p}$ attains the value $-\infty$, there is of course no hope to extend operators under domination of P to the whole space E).

b) If so, are the conditions (i), (ii), (iii) true for $\hat{p}$?

c) Is it possible to find alternative descriptions of $\hat{p}$ facilitating computations with $\hat{p}$?

The solution of these problems is the aim of the next section.

3. LOWER SEMICONTINUITY OF ISOTONE SUBLINEAR OPERATORS

<u>3.1 Lemma:</u> If F'_s is the sup-completion of the dual F' of a locally convex vector lattice F, $\varphi \in F'_s$ and $f_o \in F_+$, then $\varphi^-(f_o) = -\inf\{\varphi(g) : g \in F_+, g \leq f_o\}$.

<u>Proof:</u> It is easy to show that the functional $\psi : F_+ \to \mathbb{R}$ defined by

$$\psi(f) = \inf\{\varphi(g) : g \in F_+, g \leq f\}$$

is additive and positively homogeneous. Since $-\varphi^-(f) \leq -\varphi^-(g) \leq \varphi(g)$ for all $f, g \in F_+$ such that $g \leq f$, we conclude $-\varphi^- \leq \psi \leq 0$. Therefore, ψ is continuous at 0, hence continuous on F_+, i.e. $\psi \in F'_s$. From the obvious inequality $\psi \leq \varphi$ we thus deduce that $\psi \leq -\varphi^-$, which yields $\psi = -\varphi^-$. ∎

<u>3.2 Definition:</u> Given a locally convex vector lattice E and a normed vector lattice F a mapping $\Phi : E \to F'_s$ is *l.s.c. at* 0 iff for every $\varepsilon > 0$ there is a zero-neighborhood U in E such that $\|(Pe)^-\| \leq \varepsilon$ for all $e \in U$.

3.3 Lemma: Let P be a sublinear operator from a locally convex vector lattice E into the sup-completion F'_s of the topological dual of a normed vector lattice F. Then P is l.s.c. at 0 if and only if the bisublinear form $p : E \times F \to \mathbb{R}_\infty$ associated with P is l.s.c. at $(0,0)$ with respect to the product topology.

Proof: If P is l.s.c. at 0 and $\varepsilon > 0$, then there exists $U \in \mathfrak{U}$ such that $\|(Pu)^-\| < \varepsilon$ for all $u \in U$. Hence $p(u,f) = Pu(f) \geq -\varepsilon$ for all $u \in U$ and $f \in F_+$, $\|f\| \leq 1$. Since this inequality clearly extends to all $f \in F$, $\|f\| \leq 1$, p is l.s.c. at $(0,0)$. Conversely, if p is l.s.c. at $(0,0)$, there is $W \in \mathfrak{U}$, $\delta > 0$ such that $p(e,f) \geq -\varepsilon$ for all $e \in W$ and $f \in F$, $\|f\| \leq \delta$. Given $e \in \delta \cdot W$ we conclude $Pe(f) = P\left(\frac{1}{\delta}e\right)(\delta f) \geq -\varepsilon$ for all $f \in F_+$, $\|f\| \leq 1$. It thus follows from Lemma 3.1 that

$$\|(Pe^-)\| = \sup\{(Pe)^-(f) : f \in F_+, \|f\| \leq 1\} = -\inf\{-(Pe)^-(f) : f \in F_+, \|f\| \leq 1\}$$
$$= \inf\{Pe(g) : g \in F_+, \|g\| \leq 1\} \leq \varepsilon,$$

which shows that P is l.s.c. at 0. ∎

For monotone sublinear operators on Frechet lattices we automatically have lower semicontinuity at 0:

3.4 Proposition: Let E be a Frechet lattice, F a normed vector lattice and $P: E \to F'_s$ a monotone sublinear operator. Then P is l.s.c. at 0.

Proof: Using the inverse ordering if necessary we may assume without loss of generality that P is isotone. Suppose P were not l.s.c. at 0. Then $\{\|Pu\| : u \in U, u \leq 0\}$ is unbounded for every zero-neighborhood U in E. To show this note first that $Pu \leq 0$ for $u \leq 0$ since P is isotone. Hence $Pu \in F'$ by condition (S4) of section 1. Suppose that $\{\|Pu\| : u \in U, u \leq 0\}$ were bounded from above by $r > 0$. Given $\varepsilon > 0$ we then conclude that $\|Pu\| < \varepsilon$ for all $u \in \frac{\varepsilon}{r}U$, $u \leq 0$. If V is a solid zero-neighborhood in E contained in $\frac{\varepsilon}{r} \cdot U$, then $-v^- \in V$ and $P(-v^-) \leq -(Pv)^-$ for each $v \in V$. Hence

$\|(Pv)^-\| \leq \|P(-v^-)\| \leq \varepsilon$ for all $v \in V$ contradicting the assumption that P were not l.s.c. at 0.

Let d be a translation-invariant metric on F compatible with the Frechet space topology. For each $n \in \mathbb{N}$ we can select $u_n \in E$ such that

$d(u_n, 0) < \frac{1}{2^n}$, $u_n \leq 0$ and $\|P(u_n)\| > n$,

since $\{\|Pu\| : u \in E,\ u \leq 0,\ d(u,0) < \frac{1}{2^n}\}$ is unbounded. Since the partial sums of the series $\sum_{n=1}^{\infty} u_n$ form a Cauchy sequence, it is convergent. Moreover, $u := \sum_{n=1}^{\infty} u_n$ is a lower bound for every u_n. Therefore $\|Pu\| \geq \|Pu_n\| > n$ for all $n \in \mathbb{N}$ which is absurd. ∎

If E is a locally convex vector lattice, $\mathfrak{U}$ will always denote the fundamental system of all convex, solid neighborhoods of 0 in E.

<u>3.5 Theorem:</u> Let E be a Frechet lattice, F a normed vector lattice and $P : E \to F'_s$ an isotone sublinear operator. Then there exists a greatest sublinear operator $\hat{P} : E \to F'_s$ dominated by P such that $e \to \hat{P}e(f)$ is l.s.c. for each $f \in F_+$. $\hat{P}$ is again isotone and

$$\hat{P}e(f) = \sup_{U \in \mathfrak{U}} \inf_{u \in U} P(e+u)(f) \text{ for each } (e,f) \in E \times F_+.$$

Moreover, the functional $(e,f) \to \hat{P}e(f)$ $((e,f) \in E \times F_+)$ is l.s.c. at each point $(e,f) \in E \times F_+$ where the following condition holds:

$\exists u \in U : P(e+u)(f) < \infty$ for all $U \in \mathfrak{U}$.

<u>Proof:</u> Let $q : E \times F \to \mathbb{R} \cup \{-\infty, \infty\}$ be defined by

$$q(e,f) = \begin{cases} \sup_{U \in \mathfrak{U}} \inf_{u \in U} P(e+u)(f) & \text{if } e \in E,\ f \in F_+ \\ 0 & \text{if } e = 0, f \in F \setminus F_+ \\ \infty & \text{else.} \end{cases}$$

Since P is l.s.c. at 0, there exists $V \in \mathfrak{U}$ such that $\|(Pe)^-\| \leq 1$ for all $e \in V$. Selecting $U \in \mathfrak{U}$ such that $U + U \subset V$ we conclude $\|P(e+u)^-\| \leq 1$ for all $e, u \in U$. It follows that

$$q(e,f) \geq \inf_{u \in U} P(e+u)(f) \geq \inf_{u \in U} (-P(e+u)^-)(f) \geq -\|f\| \text{ for all } e \in U, f \in F_+.$$

This inequality obviously remains true for all $f \in F$. Furthermore, it is easy to check that q is positively homogeneous in each variable. Hence

$$q(e,f) \geq -p_U(e) \cdot \|f\| \quad \text{for all } e \in E,\ f \in F,$$

where p_U denotes the Minkowski functional of U, which implies that $q(E \times F) \subset \mathbb{R}_\infty$. Given $e \in E$, let us show that $f \to q(e,f)$ is subadditive. Since $q(e, f_1 + f_2) \leq q(e,f_1) + q(e,f_2)$ whenever $f_1 \notin F_+$ or $f_2 \notin F_+$, it suffices to prove the subadditivity on F_+.

Let $f_1, f_2 \in F_+$, $U \in \mathfrak{U}$ such that $V+V\subset U$ and $v_1, v_2 \in V$. Then

$$\inf_{u\in U} P(e+u)(f_1+f_2) \leq P(e-v_1^--v_2^-)(f_1+f_2) \leq P(e-v_1^-)(f_1) + P(e-v_2^-)(f_2)$$
$$\leq P(e+v_1)(f_1) + P(e+v_2)(f_2) .$$

Since v_1, v_2 were arbitrary, we conclude

$$\inf_{u\in U} P(e+u)(f_1+f_2) \leq \inf_{v_1\in V} P(e+v_1)(f_1) + \inf_{v_2\in V} P(e+v_2)(f_2) \leq q(e,f_1)+q(e,f_2).$$

Passing to the supremum on the left side, it follows that

$$q(e,f_1+f_2) \leq q(e,f_1) + q(e,f_2).$$

Moreover, if $U \in \mathfrak{U}$ and $u \in U$ then

$$\inf_{u_1\in U} P(e+u_1)(f_1) + \inf_{u_2\in U} P(e+u_2)(f_2) \leq P(e+u)(f_1) + P(e+u)(f_2)$$
$$= P(e+u)(f_1+f_2),$$

therefore

$$\inf_{u_1\in U} P(e+u_1)(f_1) + \inf_{u_2\in U} P(e+u_2)(f_2) \leq \inf_{u\in U} P(e+u)(f_1+f_2)$$
$$\leq q(e,f_1+f_2),$$

which implies that

$$q(e,f_1) + q(e,f_2) \leq q(e,f_1+f_2).$$

It follows that $f \to q(e,f)$ is additive on F_+.

The proof of the subadditivity in the first variable is straightforward. We claim that $q|_{E\times F_+}$ is separately l.s.c.. Given $(e,f) \in E\times F_+$ and $\alpha < q(e,f)$ there is $U \in \mathfrak{U}$ such that $\inf_{u\in U} P(e+u)(f) > \alpha$. If $V \in \mathfrak{U}$ satisfies $V+V\subset U$, then, for each $e' \in V$,

$$q(e+e',f) \geq \inf_{v\in V} P(e+e'+v)(f) \geq \inf_{u\in U} P(e+u)(f) > \alpha.$$

Hence $x \to q(x,f)$ is l.s.c. at e.

In order to show that $g \to q(e,g)$ $(g \in F_+)$ is lower semicontinuous at f, choose $U \in \mathfrak{U}$ as above. Suppose that, for each $n \in \mathbb{N}$, there exists $f_n \in F_+$ such that $\| f-f_n\| < \frac{1}{n}$ and

$$\inf_{v\in V} P(e+v)(f_n) < \alpha \qquad \text{for all } V \in \mathfrak{U}.$$

Inductively, it is possible to construct a sequence $(W_n)_{n\in\mathbb{N}}$ of closed, convex, solid neighborhoods of 0 in E such that $W_1 + W_1 \subset U$ and $W_{n+1} + W_{n+1} \subset W_n$ for all $n\in\mathbb{N}$.

Moreover, we can select elements $w_n \in W_n$ with the property $P(e+w_n)(f_n) < \alpha$. Since $P(e-w_n^-)(f_n) \leq P(e+w_n)(f_n)$ and $-w_n^- \in W_n$, we may assume that $w_n \leq 0$ for each $n\in\mathbb{N}$.

It is easy to check that the partial sums of the series $\sum_{n=1}^{\infty} w_n$ form a Cauchy sequence in E and that $w := \sum_{n=1}^{\infty} w_n \in W_1+W_1 \subset U$.

Furthermore, w is a lower bound of every w_n, hence

$$P(e+w)(f_n) \leq P(e+w_n)(f_n) < \alpha \quad \text{for all } n \in \mathbb{N}.$$

On the other hand,

$$\alpha < \inf_{u \in U} P(e+u)(f) \leq P(e+w)(f),$$

which contradicts the lower semicontinuity of $P(e+w)$ at f.
Therefore, $g \to q(e,g)$ is l.s.c. at f. It follows that q induces an isotone, sublinear operator $\hat{P} : E \to F'_s$. By definition, the functionals $e \to q(e,f)$ are the greatest l.s.c. minorants of $e \to p(e,f)$ for each $f \in F_+$, where p denotes the bisublinear functional on $E \times F$ associated with P.
Let us finally show that $(e,f) \to q(e,f)$ $((e,f) \in E \times F_+)$ is l.s.c. at (e,f) if $\forall U \in \mathfrak{U}: \exists u \in U: P(e+u)(f) < \infty$. If $(e_n, f_n)_{n \in \mathbb{N}}$ is a sequence in $E \times F_+$ converging to (e,f) then

$$q(e_n, f_n) = q(e_n, f_n - f \wedge f_n) + q(e_n, f \wedge f_n) \quad \text{for each } n \in \mathbb{N}.$$

Since $\hat{P}$ and hence also q is l.s.c. at 0 it follows that

$$\liminf_{n \to \infty} q(e_n, f_n - f \wedge f_n) \geq 0.$$

Furthermore, $f \wedge f_n \in [0,f] := \{g \in F_+ : g \leq f\}$ for all $n \in \mathbb{N}$. It therefore suffices to prove that $q|_{E \times [0,f]}$ is l.s.c. at (e,f).

Given a pair $(e,f) \in E \times F_+$ such that $q(e,f) < \infty$ and a real number $\alpha < q(e,f)$ suppose that there exists a sequence (e_n, f_n) in $E \times [0,f]$ such that $\lim_{n \to \infty} (e_n, f_n) = (e,f)$ and $q(e_n, f_n) \leq \alpha$ for all $n \in \mathbb{N}$. If d is a translation-invariant metric on F compatible with the topology on F choose $U_n \in \mathfrak{U}$, $U_n \subset \{x \in E: d(x,0) < \frac{1}{n}\}$. Let $U, V \in \mathfrak{U}$ be such that $\inf_{u \in U} P(e+u)(f) > \alpha$ and $V + V + V \subset U$. It is possible to find an element $v_0 \in V$, $v_0 \leq 0$ with the property $P(e+v_0)(f) < \infty$. Since $P(e+v_0)$ is a lower semicontinuous additive positively homogeneous functional on F_+, it is known (see [2], Lemma 3.15) that

$$\lim_{n \to \infty} P(e+v_0)(f_n) = P(e+v_0)(f)$$

or, equivalently, $\lim_{n \to \infty} P(e+v_0)(f - f_n) = 0$.
If $\varepsilon > 0$, $n_0 \in \mathbb{N}$ and $v_n \in U_n \cap V$ satisfy the following conditions

$$\alpha + \varepsilon < \inf_{u \in U} P(e+u)(f),$$

$$e_n - e \in V \text{ and } |P(e+v_0)(f-f_n)| < \varepsilon \quad \text{for all } n \geq n_0 \quad \text{and}$$

$$P(e_n+v_n)(f_n) \leq \alpha \quad \text{for all } n \in \mathbb{N},$$

then $P(e_n+v_n)(f_n) = P(e+(e_n-e)+v_n)(f_n) \geq P(e+v_0 \wedge (v_n+(e_n-e)))(f_n)$.

Since P is isotone, we have $P(e+v_0\wedge(v_n+(e_n-e)))(f) -$
$- P(e+v_0\wedge(v_n+(e_n-e)))(f_n) = P(e+v_0\wedge(v_n+(e_n-e)))(f-f_n)$

$$\leqq P(e+v_0)(f-f_n) < \varepsilon \quad \text{for all } n \geqq n_0 .$$

Therefore, $P(e_n+v_n)(f_n) \geqq P(e+v_0\wedge(v_n+(e_n-e)))(f) - \varepsilon$

$$\geqq \inf_{u\in U} P(e+u)(f) - \varepsilon > \alpha \qquad \text{for all } n \geqq n_0,$$

contradicting the assumption. It follows that q is l.s.c. at (e,f). ∎

3.6 Example: Let H be a linear subspace of a Frechet lattice E, F a normed vector lattice and let $T_0: H\to F'$ be a positive continuous operator. For a continuous linear operator $T: E\to F'$ the following statements are equivalent:

i) T is a positive extension of T_0

ii) T is positive and dominated by the sublinear operator $P_0 : E \to F'_s$, where $P_0 e = \infty$ for all $e \in E \setminus H$, $P_0 h = T_0 h$ for all $h \in H$.

iii) T is dominated by the isotone sublinear operator $P : E \to F'_s$, where $Pe = \inf\{P_0 x : x \in E,\ x \geqq e\} = \inf\{T_0 h : h \in H,\ h \geqq e\}$.

iv) T is dominated by $\hat{P} : E \to F'_s$, where
$\hat{P}e(f) = \sup_{U\in\mathfrak{U}} \inf\{(\bigwedge_i T_0 h_i)(f) : (h_i) \in H_{e,U}\}$
and $H_{e,U}$ denotes the set of all finite families $(h_i)_{i\in I}$ in H such that $(e-\bigwedge_i h_i)^+ \in U$.

Proof: It is easy to check that
$$\sup_{U\in\mathfrak{U}} \inf_{u\in U} P(e+u)(f) = \sup_{U\in\mathfrak{U}} \inf\{(\bigwedge_i T_0 h_i)(f) : (h_i) \in H_{e,U}\}.$$
The rest of the proof is obvious.

Reviewing Example 3.6 note that the sublinear operator P_0 as well as P only attain values in $F'\cup\{\infty\}$. (This is, in general, no longer true for $\hat{P}$). The advantage of constructing $\hat{P}$ from P thereby approaching the regularization $P^\cap$ (provided that $K_{P,0} \neq \emptyset$ or, equivalently, that there exists at least one continuous linear operator dominated by P) will soon become evident. As mentioned at the end of the last chapter if $P = P^\cap$ then the bisublinear functional q on $E\times F$ associated with P must be l.s.c. at each point $(e,f) \in E\times F$. It is an immediate consequence of Theorem 3.5 and of the following lemma that the lower semi-continuity of q can fail only at points $(e,f) \in E\times F_+$ where $q(e,f) = \infty$.

3.7 Lemma: Let E be a locally convex space, F a locally convex vector lattice and $p : E \times F \to \mathbb{R}_\infty$ be a bisublinear functional satisfying $\{f \in F : p(e,f) < \infty\} \subset F_+$ for each $e \in E \setminus \{0\}$. Then p is l.s.c. at every point $(e,f) \in E \times (F \setminus F_+)$. If $p|_{E\times F_+}$ is l.s.c. at $(e,f) \in E \times F_+$, then p is also l.s.c. at (e,f).

Proof: Let $(e,f) \in E \times (F \setminus F_+)$ and $\alpha < p(e,f)$ be given. Since F_+ is closed, there is a neighborhood V of f not intersecting F_+. If $e = 0$ then $\alpha < 0 = p(e,f) \leq p(x,g)$ for all $x \in E$, $g \in V$. If $e \neq 0$ then $\alpha < p(x,g) = \infty$ for all $x \in U$, $g \in V$, where U is a neighborhood of e not containing 0. Therefore, p is l.s.c. at (e,f). Finally, let $(e,f) \in E \times F_+$, $\alpha < p(e,f)$ and assume that $p|_{E\times F_+}$ is l.s.c. at (e,f). Then there are neighborhoods U of e and V of f such that $\alpha < p(x,g)$ for all $x \in U$, $g \in V \cap F_+$. It follows that $\alpha < p(x,g)$ for all $x \in U$, $g \in V$. This is evident whenever $0 \notin U$. If $0 \in U$, the inequality $\alpha < p(x,g)$ is again obvious for $x \in U \setminus \{0\}$, $g \in V$. For $x = 0$ and $g \in V$ we have $\alpha < p(x,f) = 0 = p(x,g)$. Consequently, p is l.s.c. at (e,f). ∎

3.8 Remark: Lemma 3.7 shows that it would perhaps be more natural to define the bisublinear functional associated with a sublinear operator $P : E \to F'_s$ only on $E \times F_+$. On the other hand, the subsequent use of tensor products then requires the introduction of tensor products of *cones*. Since tensor products of vector spaces are more common, we preferred to use the domain $E \times F$ for the associated bisublinear functional.

By abuse of language we shall say that a sublinear operator $P : E \to F'_s$ is l.s.c. iff the associated bisublinear functional is l.s.c..

Lemma 3.3 shows that this convention is consistent with Definition 3.2.

As indicated by Example 3.6, extension problems of positive operators can often be formulated using the domination by a sublinear operator $P : E \to F'_s$ attaining only values in $F' \cup \{\infty\}$. In this case $\hat{P}$ is l.s.c.:

3.9 Theorem: Let P be an isotone sublinear operator from a Frechet lattice E into the sup-completion F'_s of the dual F' of a normed vector lattice F. If $P(E) \subset F' \cup \{\infty\}$ and P is l.s.c. at 0 then the bisublinear functional

$$\hat{p}(e,f) = \begin{cases} \hat{P}e(f) & \text{if } e \in E,\ f \in F_+ \\ 0 & \text{if } e = 0,\ f \in F \setminus F_+ \\ \infty & \text{else} \end{cases}$$

associated with $\hat{P}$ is the greatest l.s.c. minorant of the bisublinear functional p induced by P on $E \times F$. In particular, $\hat{P}$ is l.s.c.

Proof: By Theorem 3.5 and Lemma 3.7 it suffices to show that $\hat{p}|_{E\times F_+}$ is l.s.c. at each point $(e,f) \in E \times F_+$ at which $P(e+u)(f) = \infty$ for all $u \in U$ and some $U \in \mathfrak{U}$ (hence $P(e+u) = \infty$ for all $u \in U$). Selecting $V \in \mathfrak{U}$ such that $V + V \subset U$ we conclude $P(x+v)(g) = \infty$ for all $x \in e + V$, $g \in F \setminus \{0\}$, which implies that $\hat{p}(x,g) = \infty$. It follows that $\hat{P}$ is l.s.c. The rest of the proof is an immediate consequence of Theorem 3.5. ∎

If $q \in]1,\infty[$, $F = L^q(\mu)$ for some σ-finite measure space $(\Omega, \mathbb{A}, \mu)$, then the dual F' is isometrically isomorphic to $L^{q'}(\mu)$ where $q' := \frac{q}{q-1}$. Thus the sup-completion F'_s can not only be interpreted as the cone of all l.s.c. positively homogeneous, additive $\mathbb{R}_\infty$-valued functionals on F_+ but also as the cone of all (equivalence classes of) μ-measurable functions from Ω into $\mathbb{R}_\infty$ minorized by some function in $L^{q'}(\mu)$. Similarly, although $L^1(\mu)$ is not a dual space in general, it is at least a KB-space in the sense of Vulikh (see [6]), i.e. a band in its bidual. It can be shown that the sup-completion of a KB-space G can also be interpreted as a subcone of the sup-completion of G" (see [2], Thm. 7.12). Unter this point of view an element of G_s is a l.s.c. positively homogeneous, additive $\mathbb{R}_\infty$-valued functional φ on G'_+ that satisfies the order-continuity condition $\lim_{n\to\infty} \varphi(g'_n) = \varphi(g')$ for every increasing sequence (g'_n) in G'_+ with $g' = \sup_{n\in\mathbb{N}} g'_n \in G'_+$. If $J: G_s \to G''_s$ denotes the natural imbedding and P is an isotone sublinear operator from a Frechet lattice E into G_s, then $J \circ P$ is an isotone sublinear map from E into G''_s so is $\widehat{J\circ P}$. Provided that the range of $\widehat{J\circ P}$ is contained in the natural image $J(G_s)$ of G_s in G''_s, we shall be able to reformulate the majority of results about extensions of positive operators mapping into L^p-spaces, $p > 1$, also in the context of L^1-spaces (see [2]). Thus we are strongly interested in an alternative description of $\hat{P}$ that allows the direct interpretation of the values $\hat{P}e$ as μ-measurable functions. Remarkably, it is the *equivalence of the order-topology and the original locally convex topology in Frechet lattices* that leads to a solution of this problem. To show this we need the following

3.10 Definition: Let E be a Frechet lattice, F a normed vector lattice and $P : E \to F'_s$ an isotone sublinear operator. If $a \in E_+$ we put

$$\hat{P}^a e(f) = \sup_{\varepsilon>0} \inf_{|u|\leq\varepsilon a} P(e+u)(f) \qquad \text{for all } f \in F,\ e \in E.$$

It is easy to show that $\hat{P}^a : E \to F'_s$ is again sublinear and isotone.

3.11 Theorem: Let P be an isotone, sublinear operator from a Frechet lattice E into the sup-completion F'_s of the dual of a normed vector lattice F. Then

$$\hat{P}e(f) = \inf_{a\in E_+} \hat{P}^a e(f) \qquad \text{for each } e \in E,\ f \in F'_+.$$

Proof: For each $U \in \mathfrak{U}$ and each $a \in E_+$ there exists $\varepsilon > 0$ such that $\{u \in E : |u| \leq \varepsilon a\} \subset U$. Hence $\hat{P}e(f) \leq \inf_{a\in E_+} \hat{P}^a e(f)$ for all $e \in E$, $f \in F'_+$.

Suppose, that there were elements $e \in E$, $f \in F_+$ such that $\hat{P}e(f) < \inf_{a\in E_+} \hat{P}^a e(f)$. Choose $\alpha \in \mathbb{R}$ with $\hat{P}e(f) < \alpha < \inf_{a\in E_+} \hat{P}^a e(f)$.
For each $a \in E_+$, let $M_a = \{r > 0 : \inf_{|u|\leq \frac{1}{r}a} P(e+u)(f) \geq \alpha\}$. From the choice of α we conclude that $M_a \neq \emptyset$.
The functional $\varphi : E \to \mathbb{R}_\infty$ defined by $\varphi(a) = -\inf M_a$ is obviously decreasing. It is easy to check that $\lambda M_a = M_{\lambda a}$ for all $\lambda > 0$, $a \in E_+$. Since $M_0 =]0,\infty[$, φ is positively homogeneous. If $a_1, a_2 \in E$ are such that $\varphi(a_1) \neq 0$, $\varphi(a_2) \neq 0$, we claim that $M_{a_1+a_2} \subset M_{a_1} + M_{a_2}$. To prove this inclusion note first that M_a is an interval in $\mathbb{R}$ unbounded from above. Therefore, if $s > 0$ is such that $s \notin M_{a_1} + M_{a_2}$, there are real numbers $r_1, r_2 > 0$, $s = r_1 + r_2$, not contained in M_{a_1} and M_{a_2}, respectively, i.e.

$$\inf_{r_1\cdot|u|\leq a_1} P(e+u)(f) < \alpha \quad \text{and} \quad \inf_{r_2\cdot|u|\leq a_2} P(e+u)(f) < \alpha\ .$$

Hence there are elements $u_1, u_2 \in E$ satisfying $r_1 \cdot |u_1| \leq a_1$, $r_2 \cdot |u_2| \leq a_2$, $P(e+u_1)(f) < \alpha$ and $P(e+u_2)(f) < \alpha$. Setting

$$u := \frac{r_1}{r_1+r_2} u_1 + \frac{r_2}{r_1+r_2} u_2 \in E$$

we thus conclude $(r_1+r_2)\cdot|u| = |r_1 u_1 + r_2 u_2| \leq a_1 + a_2$ and

$$\begin{aligned} P(e+u)(f) &= P\Big(\frac{r_1}{r_1+r_2}(e+u_1) + \frac{r_2}{r_1+r_2}(e+u_2)\Big) \\ &\leq \frac{r_1}{r_1+r_2} P(e+u_1) + \frac{r_2}{r_1+r_2} P(e+u_2) < \alpha, \end{aligned}$$

which implies that $r_1 + r_2 \notin M_{a_1+a_2}$.

We have thus proved the inclusion $M_{a_1+a_2} \subset M_{a_1} + M_{a_2}$, which implies that φ is subadditive. If $\tilde{\varphi} : E \to \mathbb{R}_\infty$ is defined by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in E_+ \\ \infty & \text{else} \end{cases}$$

then $\tilde{\varphi}$ is sublinear. Moreover, using the same arguments as in the proof of Proposition 3.4 it can be shown that $\tilde{\varphi}$ is l.s.c. at 0, since

φ is decreasing. Let ℓ be a continuous linear form on E dominated by $2\cdot\tilde{\varphi}$. Since $2\cdot\tilde{\varphi}$ is decreasing, we have $\ell(x) \leq 2\cdot\tilde{\varphi}(x) \leq 2\cdot\tilde{\varphi}(0) = 0$ for all $x \in E_+$, hence $\ell_o := -\ell$ is positive. Setting

$$U_a := \begin{cases} \{x \in E : \ell_o(a)\cdot|x| \leq a\} & \text{if } \ell_o(a) \neq 0 \\ \bigcup_{\lambda>0} [-\lambda a, \lambda a] & \text{else} \end{cases}$$

for each $a \in E_+$, the convex hull U of $\bigcup_{a\in E_+} U_a$ is a zero-neighborhood in E with respect to the order topology. Since E is a Frechet lattice, U is in fact also a zero-neighborhood for the original topology on E. We claim that $U \subset \bigcup_{a\in E_+} U_a$. Once this inclusion is proved it follows that $\inf_{u\in U} P(e+u)(f) \geq \inf_{a\in E_+} \inf_{u\in U_a} P(e+u)(f)$. Moreover, if $\ell_o(a) > 0$ then $\ell_o(a) > -\varphi(a)$ for each $a \in E_+$. This inequality being trivial for $\varphi(a)=0$ it results from $\ell_o(a) \geq -2\cdot\tilde{\varphi}(a) = -2\cdot\varphi(a) > -\varphi(a)$ for $\varphi(a) < 0$. Consequently, $\ell_o(a) \in M_a$ in this case, which implies that $\inf_{u\in U_a} P(e+u)(f) \geq \alpha$. If $\ell_o(a) = 0$, then $U_a = \bigcup_{\lambda>0} [-\lambda a, \lambda a]$ i.e. for every $u \in U_a$ there exists $r > 0$ such that $r\cdot|u| \leq a$. On the other hand, $-\varphi$ being majorized by ℓ_o on E_+ φ must also vanish at a. Therefore $r \in M_a$ and $P(e+u)(f) \geq \alpha$. We thus have the estimate $\inf_{u\in U_a} P(e+u)(f) \geq \alpha$ for every $a \in E_+$, which yields $\alpha \leq \inf_{u\in U} P(e+u)(f) \leq \sup_{U\in\mathfrak{U}} \inf_{u\in U} P(e+u)(f)$, since $\mathfrak{U}$ is a neighborhoodbasis of 0. This contradicts the choice of α. To complete the proof it thus remains to show that $U \subset \bigcup_{a\in E_+} U_a$.
If $x \in U$ then there exist finite families $(x_a)_{a\in A}$ in E, $(\lambda_a)_{a\in A}$ in $[0,1]$ and $(r_a)_{a\in A} \in]0,\infty[$ such that $\sum_{a\in A} \lambda_a = 1$, $x = \sum_{a\in A} \lambda_a x_a$, $r_a\cdot\ell_o(a) \leq 1$ and $|x_a| \leq r_a\cdot a$ for all $a \in A$.
Setting $a_x := \sum_{a\in A} \lambda_a r_a\cdot a \in E_+$ we obtain $\ell_o(a_x) = \sum_{a\in A} \lambda_a r_a \ell_o(a) \leq \sum_{a\in A} \lambda_a = 1$, hence $|x|\cdot\ell_o(a_x) \leq \sum_{a\in A} \lambda_a |x_a| \leq \sum_{a\in A} \lambda_a\cdot r_a\cdot a = a_x$ which implies that $x \in U_{a_x}$. ∎

<u>3.12 Corollary:</u> Let P be an isotone sublinear operator from a Frechet lattice into the sup-completion of the dual F' of a normed vector lattice F. For each $e \in E$, $f \in F_+$ there is an element $a \in E_+$ such that $\hat{P}e(f) = \hat{P}^a e(f)$.

<u>Proof:</u> It follows immediately from the definition of $\hat{P}^a$ that $\hat{P}^a \leq \hat{P}^b$ whenever $a, b \in E_+$ satisfy $b \leq \lambda a$ for some $\lambda \geq 0$. If $(a_n)_{n\in\mathbb{N}}$ is a sequence in $E_+ \setminus \{0\}$ and d denotes a translation invariant metric on E, choose $\lambda_n > 0$ for each $n \in \mathbb{N}$ such that $d(\lambda_n a_n, 0) < \frac{1}{2^n}$.

Then $\lambda_n a_n \leq a := \sum_{m=1}^{\infty} \lambda_m a_m$, which yields $\hat{P}^a \leq \hat{P}^{a_n}$ for every n.
Given $e \in E$, $f \in F_+$, it follows from Theorem 3.11 that we can select $a_n \in E_+$ such that $\hat{P}^{a_n}e(f) \leq \hat{P}e(f) + \frac{1}{n}$ for each $n \in \mathbb{N}$. If a is constructed as above then $\hat{P}^a e(f) \leq \hat{P}e(f)$. Since always $\hat{P}^a e(f) \geq \hat{P}e(f)$ equality results.

Returning to the problems mentioned before Definition 3.10 let $(\Omega, \mathbb{A}, \mu)$ be a σ-finite measure space, $q \in [1,\infty[$, and C^q be the cone of all (equivalence classes of) μ-measurable $\mathbb{R}_\infty$-valued functions minorized by some element of $L^q(\mu)$. Recall that C^q is ordered cone isomorphic to the sup-completion of $L^q(\mu)$. If P is an isotone sublinear operator from a Frechet lattice E into C^q then $\inf_{|u| \leq \varepsilon a} P(e+u)$ is a function in C^q. Similarly, $\hat{P}^a e$ as well as $\inf_{a \in E} \hat{P}^a e$ are directly seen to be members of C^q. If $q = 1$ and J is the natural imbedding of $L^1(\mu)_s$ into $L^1(\mu)''_s$, we thus see that $\widehat{J \circ P}$ only attains values in C^1. (For the details of this argument see [2], p. 137 ff.)

3.13 Remark: Using Lemma 1.1 and the equivalence of relative uniform *-convergence (see [4], Ch. IV, § 2) and the topological convergence in Frechet lattices it can be shown that, for each $e \in E$ there exists an element $a \in E_+$ such that $\hat{P}e = \hat{P}^a e$. Since we do not need this fact here, we omit the proof.

4. APPLICATIONS TO POSITIVE OPERATOR EXTENSION

In this section we shall be exclusively concerned with the following situation: E is a locally convex vector lattice, H a linear subspace of E, F a topological vector lattice that will always be assumed to be Dedekind complete and $T: H \to F$ will be a positive continuous operator. Our aim is to construct positive (continuous) extensions of T and to describe the set of all extensions.
If $e \in E \setminus H$ and $\{Th : h \geq e\}$ or $\{Th : h \geq -e\}$ is not bounded from below in F, then there cannot be any positive extension S of T with domain including e since otherwise

$$Th \geq Se \quad \text{for all } h \in H,\ h \geq e \quad \text{and}$$
$$Th \geq -Se \quad \text{for all } h \in H,\ h \geq -e.$$

Thus we should at least postulate that $\{Th : h \geq e\}$ is bounded from below in F for all $e \in E$, or, equivalently, that T(A) is bounded from below in F for every subset A of H bounded from below in E. Under this condition the first step to a solution of the positive extension problem is the construction of the isotone sublinear operator $P:E\to F_s$ given by $Pe = \inf\{Th: h \geq e\}$ for all $e \in E$. While the concept of forming $\hat{P}$ via Theorem 3.5 is available only for duals of normed vector lattices or at least KB-spaces, the sublinear operators $\hat{P}^a$ $(a \in E_+)$ are well-defined with values in F_s even in the context of arbitrary Dedekind complete vector lattices. Moreover, if

$$E_a := \{x \in E: \exists\, \lambda > 0: \ |x| \leq \lambda a\}$$

is the vector lattice ideal generated by a, then P^a is already regularized on $H + E_a$ under a suitable locally convex topology:

<u>4.1 Theorem:</u> Given $a \in E_+$, let $\mathfrak{P}_a$ denote the set of all positive extensions of T to $H+E_a$. If T(A) is bounded from below for every subset A of H bounded from below in E, then

$$P^a e = \sup\{Se : S \in \mathfrak{P}_a\} \quad \text{for all } e \in H+E_a.$$

<u>Proof:</u> Note that $\hat{P}^a e = \sup_{\varepsilon>0} \inf\{Th: h \in H, h \geq e-\varepsilon a\}$. Since $\{h \in H: h \geq e-\varepsilon a\}$ is bounded from below in E $\inf\{Th : h \in H, h \geq e - \varepsilon\cdot a\}$ exists in F for every $\varepsilon > 0$, consequently, $\hat{P}^a e \in F_s$. A routine argument shows that $\hat{P}^a$ is isotone and sublinear and that $\hat{P}^a h = Th$ for all $h \in H$.
Let H_1 be an algebraic complement of E_a in $E_a + H$ such that $H_1 \subset H$. If m_a denotes the Minkowski functional of the order interval $[-a,a]$ in E_a and q is an arbitrary continuous seminorm on E, then

$$s_q(h_1 + x) = q(h_1) + m_a(x) \qquad (h_1 \in H_1,\ x \in E_a)$$

defines a seminorm s_q on $H+E_a$.
For every $r > 0$, let $U_r := \{h_1+x : h_1 \in H_1, m_a(x) < r\}$. Given $\varepsilon > 0$ note that the affine linear mapping $k_\varepsilon: e+U_{\varepsilon/2} \to F$ with values $k_\varepsilon(e+h_1+x) = \inf\{Th : h \in H, h \geq e-\varepsilon a\} + Th_1$ $(h_1 \in H_1, x \in E_a, m_a(x) < \frac{\varepsilon}{2})$ is majorized by $\hat{P}^a$ on $e + U_{\varepsilon/2}$. Indeed, for each $h_1 \in H_1$, $x \in E_a$ such that $m_a(x) < \frac{\varepsilon}{2}$ we have

$$\begin{aligned} \inf\{Th : h \in H, h \geq e-\varepsilon a\} + Th_1 &= \inf\{T(h+h_1): h \in H, h \geq e-\varepsilon a\} \\ &= \inf\{T(h') : h' \in H, h' \geq e+h_1-\varepsilon a\} \\ &\leq \inf\{T(h') : h' \in H, h' \geq e+x+h_1-\tfrac{\varepsilon}{2} a\} \\ &\leq \hat{P}^a(e+x+h_1). \end{aligned}$$

Moreover, since T is continuous on H k_ε is obviously continuous with respect to the locally convex topology $\mathfrak{T}$ induced by the seminorms s_q on $H+E_a$, where q varies over all continuous seminorms on E.
Finally $\sup_{\varepsilon>0} k_\varepsilon(e) = \hat{P}^a(e)$.
Since $e \in E_a+H$ was arbitrary, $\hat{P}^a$ is regularized with respect to $\mathfrak{T}$.

It is easy to check that a linear operator $S: H+E_a \to F$ is dominated by $\hat{P}^a$ if and only if it belongs to $\mathfrak{P}$. Furthermore the linear maps in $\mathfrak{P}$ are automatically $\mathfrak{T}$-continuous. Hence the assertion follows from the general theory outlined in section 1 (see [2], 2.12). ∎

4.2 Corollary: Under the assumptions fixed before Theorem 4.1 the following statements are equivalent:

i) For every $a \in E$ T has a positive extension to $H+\mathbb{R}a$.

ii) For every $a \in E_+$ T has a positive extension to $H+E_a$.

iii) For every subset $A \subset H$ bounded from below in E T(A) is bounded from below in F.

Proof: (ii) ⇒ (i) is trivial, (iii) ⇒ (ii) is clear from Thm. 4.1.
(i) ⇒ (iii): If $A \subset H$ is bounded from below by $a \in E$ choose a positive extension T_a of T to $H+\mathbb{R}a$. Then T(A) is bounded from below by $T_a(a)$.∎

In order to extend a positive operator to the whole space E the relationship between the orderings of E and F cause overwhelming troubles, in general. At least in classical Banach lattices, however, we can procede further using the results of [2]:
Although every pair (E,G) of adapted Banach lattices in the terminology of [2] could replace the (L^p, L^q)-configuration introduced now, we only deal with this more restrictive situation, since the central ideas are not hidden behind too much technical framework in this case. Thus, from now on, let E always be an L^p-space, G an L^q-space with $1 < q \leq p < \infty$. Setting $F := G'$ we may identify F' with G since G is reflexive. It was shown in [2] that in this context an isotone sublinear map from E into $F'_s = G_s$ is regularized if and only if it is l.s.c.. As an immediate consequence of the results prepared in the last section we hence reprove some deep extension theorems in L^p-spaces.

4.3) Theorem: The positive operator $T: H \to G$ has a positive extension $T_0: E \to G$ if and only if T(A) is bounded from below for every subset $A \subset H$ bounded from below in E (see [2], Thm. 4.7).

Proof: For the sublinear operator P mentioned above $\hat{P}$ is regularized.∎

4.4) Theorem: Assume that T has a positive extension. Given $e \in E$, $f \in F$ and $\alpha \in \mathbb{R}$ such that

$$\inf_{\varepsilon>0}(\sup\{Th: h \in H, h \leq e+\varepsilon a\})(f) < \alpha < \sup_{\varepsilon>0}(\inf\{Th: h \in H, h \geq e-\varepsilon a\})(f)$$

for all $a \in E_+$, there is a positive extension $S: E \to G$ of T such that $Se(f) = \alpha$.

Proof: The inequality is nothing else but $\alpha < \inf_{a \in E_+} \hat{P}^a e(f) = \hat{P}e(f)$ and $-\alpha < \inf_{a \in E_+} \hat{P}^a(-e)(f) = \hat{P}(-e)(f)$. Hence the theorem follows from [2], 5.9.∎

5. APPENDIX

5.1 Theorem: Let E be a vector space, F a locally convex vector lattice and let $p : E \times F \to \mathbb{R}_\infty$ be a bisublinear functional satisfying the conditions

i) $\{f \in F : p(e,f) < \infty\} \subset F_+$

ii) $f \to p(e,f)$ is additive and l.s.c. on F_+ for each $e \in E$.

Then $p(e,f) = p^{\otimes}(e \otimes f)$ for all $(e,f) \in E \times F_+$. For the proof we need the following

5.2 Lemma: Given a vector lattice F and a non-empty finite subset A of F_+, let $a \in F_+$ be such that A is contained in the vector lattice ideal

$$F_a := \{f \in F : \exists\lambda \geq 0 : |f| \leq \lambda a\}$$

generated by a. Then, for each $n \in \mathbb{N}$, there exists a finite-dimensional vector sublattice F_n of F_a and a positive projection $P_n : F_a \to F_n$ such that $a \in F_n$ and

$$f - \frac{1}{n} a \leq P_n f \leq (1 + \frac{1}{n}) f.$$

Proof: Since F_a is a vector lattice with order unit a there exists a vector lattice isomorphism $f \to \bar{f}$ from F_a onto a dense vector sublattice of some space $C(K)$, K compact, such that $\bar{a} = 1$ (see [1], p. 76).

If $n \in \mathbb{N}$, and $x \in K$ choose an open neighborhood U_x of x in K such that

$$\sup \bar{f}(U_x) - \inf \bar{f}(U_x) \leq \frac{1}{5n} \text{ and}$$

$$\bar{f}(x) \leq (1 + \frac{1}{n}) \cdot \inf \bar{f}(U_x)$$

for all $f \in A$. Then there is a finite subset $\{x_1,\dots,x_m\} \subset K$ such that $\bigcup_{i=1}^{m} U_{x_i} = K$. Without loss of generality we may assume that $x_i \in U_{x_i} \setminus \bigcup_{j \neq i} U_{x_j}$ for $i = 1,\dots,m$. Let $(g_i)_{1 \leq i \leq m}$ be a partition of unity in $C(K)$ subordinate to $(U_{x_i})_{1 \leq i \leq m}$ and choose $\delta > 0$, $5 \cdot nm \cdot \|f\| \cdot \delta \leq 1$ for all $f \in A$. Since $\{\bar{f} : f \in F_a\}$ is dense in $C(K)$, we can select $f_1,\dots,f_m \in F_a$, such that $\bar{f}_i(x_j) = \delta_{ij}$ for $i,j=1,\dots,m$,

$$g_i - \delta \leq \bar{f}_i \leq g_i + \delta \text{ and } f_i \geq 0$$

for $i = 1, 2,\dots,m$. The linear subspace F_n of F_a generated by $\{f_i : 1 \leq i \leq m\}$ is a vector sublattice of F_a. Indeed, the mapping $\Phi : \mathbb{R}^m \to F_n$ given by

$$\Phi(\alpha_1,\dots,\alpha_m) = \sum_{i=1}^{m} \alpha_i \cdot f_i$$

is an algebraic isomorphism satisfying $\Phi(\mathbb{R}^m_+) \subset F_{n+}$ (coordinatewise order on $\mathbb{R}^n$) with order-preserving inverse

$$f \to \left(\frac{f(x_1)}{f_1(x_1)}, \ldots, \frac{f(x_m)}{f_m(x_m)} \right) .$$

Hence, Φ is a vector lattice isomorphism.

Consider the projection $P_n : F_a \to F_n$ given by

$$P_n f = \sum_{i=1}^{m} \bar{f}(x_i) \cdot f_i .$$

If $\xi \in K$ and $f \in A$ there exists $\ell \in \{1,\ldots,m\}$ such that $\xi \in U_{x_\ell}$. Keeping in mind that $g_i(\xi) \neq 0$ implies $\xi \in U_{x_i}$ we conclude

$$|\bar{f}(\eta) - \bar{f}(x_i)|\, g_i(\xi) \leq \big(|\bar{f}(\eta) - \bar{f}(x_\ell)| + |\bar{f}(x_\ell) - \bar{f}(\xi)| + |\bar{f}(\xi) - \bar{f}(x_i)| \big) \cdot g_i(\xi) \leq \frac{3}{5n} \cdot g_i(\xi)$$

for all $\eta \in U_{x_\ell}$. Since $\sum_{i=1}^{m} g_i(\xi) = 1$, it follows that

$$|\bar{f}(\eta) - \overline{P_n f}(\xi)| \leq \sum_{i=1}^{m} |\bar{f}(\eta)\, g_i(\xi) - \bar{f}(x_i)\, \bar{f}_i(\xi)| \leq \sum_{i=1}^{m} |\bar{f}(\eta) - \bar{f}(x_i)|\, g_i(\xi) + \sum_{i=1}^{m} |\bar{f}(x_i)|\, |g_i(\xi) - \bar{f}_i(\xi)| \leq \frac{3}{5n} + \delta\, m\, \|f\| \leq \frac{4}{5n} .$$

We thus obtain

$$\bar{f}(\xi) - \frac{\overline{a(\xi)}}{n} \leq \sup \bar{f}(U_{x_\ell}) - \frac{1}{n} \leq \inf \bar{f}(U_{x_\ell}) - \frac{4}{5n} \leq \overline{P_n f}(\xi)$$
$$\leq \bar{f}(x_\ell) + \frac{4}{5n} \leq \left(1 + \frac{1}{n}\right) \inf \bar{f}(U_{x_\ell}) \leq \left(1 + \frac{1}{n}\right) \bar{f}(\xi),$$

which yields

$$f - \frac{a}{n} \leq P_n f \leq 1 + \frac{1}{n} f . \qquad \blacksquare$$

Using the lemma and an obvious generalization of [2], Lemma 3.15, we can now complete the proof of the theorem with exactly the same arguments as in [2], Thm. 3.17.

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ON MATRIX ORDER AND CONVEXITY

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In the first part we review some ideas and results on matrix order and convexity. For further results we refer to the survey articles of Størmer [22], Effros [10] and Tomiyama [24]. In the second part we explain a proof of extension theorems of Hahn-Banach type for operator valued linear maps into $B(H)$ based on a non-commutative analogue of the Riesz separation property valid for $B(H)$.

1. REPORT

In the theory of non-commutative operator algebras on Hilbert space the classical methods of functional analysis based on convexity theory are in general not strong enough to prove interesting results. Experience shows that one obtains better results by strengthening the notions of convex set, convex cone, order, bounded or positive linear map, convex functional in a specific way. One uses sets of matrices and inequalities for matrices over the algebra.

If A is an operator algebra, $M_n = M_n(\mathbb{C})$ the complex $n \times n$-matrices $(n \in \mathbb{N})$ then

$$A_n := M_n(A) := A \otimes_{\mathbb{C}} M_n$$

is an operator algebra with respect to matrix multiplication. So A_n has a positive cone A_n^+ and a unit ball S_n. For a linear map T on A we denote by

$$T_n := T \otimes \mathrm{id}_{M_n} : [x_{ij}] \longmapsto [T(x_{ij})]$$

the n-multiplicity map. T is called completely positive (abbreviated : cp) if all T_n $(n \in \mathbb{N})$ are positive and T is called completely bounded (abbreviated : cb) if

$$\|T\|_{cb} := \sup_{n \in \mathbb{N}} \|T_n\| < \infty .$$

The theory started with Stinespring's representation theorem [21] for completely positive linear maps $T : A \longrightarrow B(H)$. This theorem generalizes the Gelfand-Naimark-Segal representation for positive linear forms to completely positive operator valued maps. Then Arveson [3] generalized the Krein-Rutman extension theorem for positive linear forms to an extension theorem for completely positive linear maps into $B(H)$. A lot of interesting maps in the theory of operator algebras are automatically completely positive. Tomiyama [23] proved that a projection of norm one onto a subalgebra is automatically an expectation or equivalently a completely positive projection. We refer the reader to the survey of Størmer [22] for further results in this direction.

Choi and Effros [7], [10] introduced matrix ordered spaces as the right setup to study complete positivity and Arveson's extension property. In [27] we adapted the notion of a sublinear functional to matrix order and proved a corresponding matricial Hahn-Banach theorem. In [28] we defined matrix normed spaces as the appropriate spaces on which the completely bounded maps live and proved extension theorems for completely bounded maps. The structure behind all these is the following:

1.1 DEFINITION : Let E be a $\mathbb{K}$-vectorspace ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $C_n \subset E \otimes_{\mathbb{R}} (M_n)_h$ $(n \in \mathbb{N})$ a family of subsets. We call $(C_n)_{n\in\mathbb{N}}$ a matrix convex family, if the following holds:

Let $x_i \in C_{n_i}$, $\gamma_i \in M_{n_i,m}$, $i = 1,\ldots,k$, $k,n_i,m \in \mathbb{N}$ such that

$$\sum_{i=1}^{k} \gamma_i^* \gamma_i = 1_m \quad \text{then} \quad \sum_{i=1}^{k} \gamma_i^* x_i \gamma_i \in C_m .$$

In this connection $(M_n)_h$ denotes the hermitian part of the complex $n \times n$-matrices and $M_{n,m}$ the $n \times m$-matrices. Notice that $E \otimes_{\mathbb{R}} (M_n)_h = E \otimes_{\mathbb{C}} M_n =: M_n(E)$ is a $\mathbb{C}$-vector space, if E is a $\mathbb{C}$-vector space and if E is a real vector space then $E \otimes_{\mathbb{R}} (M_n)_h := M_n(E)_h$ is the hermitian part of its complexification with respect to the natural involution.

It is easy to see that all C_n are convex sets if the family $(C_n)_{n\in\mathbb{N}}$ is matrix convex. If all C_n are cones, $\mathbb{K} = \mathbb{R}$, then we speak of a matrix order on E. Examples of matrix order are the positive cones A_n^+ of operator algebras A, the dual cones and the cones of the non-commutative L^p spaces. Examples of matrix convex sets are the unit balls S_n of an operator algebra or the family of n-states

$$\mathfrak{s}_n = \left\{f : A \longrightarrow M_n \ , \quad f \ \text{ completely positive}, \quad f(1_A) = 1_n\right\}$$

of a unital operator algebra. The family of completely positive linear maps $CP(A,B_n)$ between operator algebras A,B is a family of matrix convex cones. In [20] Schmitt and the author proved that the Hilbert-Schmidt operators between two non-commutative L^2-spaces matrix ordered by the completely positive cones are completely order isomorphic to a L^2-space, which is the matrix ordered tensor product of the two L^2-spaces. From this one can easily deduce that for non-commutative L^2-spaces the maximal and the minimal matrix order of their tensor product coincide. The maximal and the minimal matrix order of the tensor product of matrix ordered spaces was studied by Choi and Effros [7], [10]: a C*-algebra A is nuclear if and only if the maximal and the minimal order coincide on $A \otimes B$ for all C*-algebras B.

If E is a locally convex vector lattice there is only one matrix order with closed cones on E. A C*-algebra A is a lattice if and only if it is commutative. For non-commutative C*-algebras A the situation is quite different. A and the opposite algebra A^{op} (the order of multiplication is reversed in A^{op}) have the same positive cone $A^+ = (A^{op})^+$, but the n-cone $M_n(A^{op})^+$ is the transposed of $M_n(A)^+$. A is commutative if and only if $M_2(A)^+ = M_2(A^{op})^+$. Choi proved [5]: a bijection $T : A \longrightarrow B$ between C*-algebras is a *-isomorphism if and only if T_2 and T_2^{-1} are positive. The question is what further informations on A are hidden in the family of cones $M_n(A)^+$ and how to use these informations.

An old question of Kadison [13] is to characterize C*-algebras in terms of order. Since the order determines only the Jordan structure and not the algebra structure one needs an additional information which is called "orientation". There are several proposals to define an oriented ordered space. Using quite different types of orientation Connes [8] and Alfsen-Hanche Olsen-Shultz [2] gave characterizations. Werner [25] and Schmitt-Wittstock [20] gave characterizations in terms of matrix ordered spaces. All these characterizations use as a common axiom a "projection property" which says: to every closed face there is a special type of positive projection of the positive cone onto this face.

We are searching for other characterizations which use generalizations of the lattice axioms or generalizations of the Riesz property. One has to be careful, both the lattice and the Riesz property characterize commutative algebras, and purely non-commutative algebras (with trivial center) are anti-lattices. We will discuss such a property, called the matricial Riesz property which characterizes injective von Neumann algebras (for example B(H)). Injectivity means that the matrix ordered space has Arveson's extension property. By a result of Choi and Effros [7] an injective unital matrix ordered space is completely order isomorphic to an injective C*-algebra. In [27] we gave a more general extension theorem for operator valued linear maps into an injective C*-algebra which generalizes the Hahn-Banach extension theorem for maps into an order complete vector lattice (cf.[9], [17]) and gave an internal characterization of injective von Neumann algebras.

With the aid of this matricial Hahn-Banach theorem it is possible to extend completely bounded linear maps T [28] and to decompose them $T = T_1 - T_2$ into two completely positive maps T_1, T_2 such that

$$\|T_i\|_{cb} = \|T_i\| \leq \|T_1 + T_2\| = \|T\|_{cb} \qquad (i = 1, 2).$$

Later on Paulsen [15] gave a proof based on Arveson's extension theorem and explained the connection of this decomposition and the solution of the similarity problem. It is possible to deduce the matricial Hahn-Banach theorem from Arveson's extension theorem and vice versa just in the same way as the Krein-Rutman extension theorem for positive linear forms is equivalent to the scalar Hahn-Banach theorem. The proof of Arveson's extension theorem is simple in the case where the completely positive map is defined on a subalgebra (Lance [14]). In chapter 2.3 we give a short proof of Arveson's extension theorem based on the matricial Riesz separation property for B(H).

In a recent preprint [11] Haagerup proves that a von Neumann algebra N is injective if and only if $CB(N,N)$ is the span of $CP(N,N)$. It is essential that N is a von Neumann algebra, because Huruya [12] gave an example of a non-injective C*-algebra B such that every completely bounded $T : A \longrightarrow B$, A a C*-algebra, is the difference of two completely positive maps.

$C(\Omega)$ has the $1+\varepsilon$-extension and decomposition property for compact linear maps. Saar [16] showed the same for completely compact linear maps into nuclear C*-algebras A : a completely bounded map T into A is called completely compact (abbreviated cc) if for $\varepsilon > 0$ there is a finite dimensional subspace $F \subset A$ such that

$$\operatorname{dist}(T_n(x), M_n(F)) < \varepsilon \quad \text{for all } \|x\| \leq 1, \quad n \in \mathbb{N} .$$

A completely positive compact map $T : B \longrightarrow C(H)$, B a C*-algebra, is always completely compact. But there is a completely bounded compact map $S : B(H) \longrightarrow C(H)$, which is not completely compact.

As a last feature I want to mention the left matrix multiplier algebra of a matrix ordered space. In the case of an order unit space it was introduced by Werner [25], Schmitt and the author gave the general definition [20]. It is an analogue to the ideal center which Wils introduced for ordered locally convex vector spaces [26]. For a C*- algebra the left matrix multiplier algebra coincides with the left multiplier algebra. For a non commutative L^2 space the left matrix multiplier algebra is a von Neumann algebra M in standard form. Schmitt and the author [20] characterized von Neumann algebras in this way.

In the recent report of Tomiyama [24] the reader will find a lot of further related results. I hope to convince the reader that matrix order and matrix convexity is a useful enrichment for the theory of operator algebras.

2.1 SOME PRELIMINARY REMARKS ON MATRIX ORDER AND INEQUALITIES

To simplify the presentation we concentrate on B(H) or C*-algebras and their subspaces with the natural order of operators. For the technique of general matrix ordered spaces we refer to Choi and Effros [7] and the report of Effros [10]. $M_n(B(H)) = B(H^n)$ is ordered by the cone of positive operators. Let $\gamma \in M_{m,n}$ be a $m \times n$ scalar matrix. The linear map

$$(2.1) \qquad M_m(B(H))^+ \ni x \longmapsto \gamma^* x \gamma \in M_n(B(H))^+$$

is completely positive. To represent sums of those maps the following notation is useful. Let $\alpha \in M_m(M_n)$, $\alpha = [\alpha_{ij}]_{i,j=1}^m$, $\alpha_{ij} \in M_n$, $x = [x_{ij}]_{i,j=1}^m \in M_m(B(H))$.

We denote the contraction of x and α by

(2.2) $$x \times \alpha = \sum_{i,j=1}^{m} x_{ij} \otimes \alpha_{ij} \quad .$$

This multiplication by block matrices is a natural analogue of the multiplication by scalars. Let $\varepsilon_{i,j}$ be the usual matrix units in M_m, $\varepsilon = [\varepsilon_{ij}]_{ij=1}^{m} \in M_m(M_m)$. Since $\varepsilon^2 = m\varepsilon$, ε is positive. ε gives the identity map: $x \times \varepsilon = x$ and hence $\gamma^* x\, \gamma = \gamma^*(x \times \varepsilon)\gamma = x \times (\gamma^* \varepsilon \gamma)$ Choi [6] showed that the map $x \longrightarrow x \times \alpha$ is completely positive if and only if α is positive. Then there exists a finite number $\gamma_1, \dots, \gamma_n \in M_{m,n}$ such that $\alpha = \sum_{\varkappa=1}^{k} \gamma_\varkappa^* \,\varepsilon\, \gamma_\varkappa$ and $x \times \alpha = \sum_{\varkappa=1}^{k} \gamma_\varkappa^* \, x \, \gamma_\varkappa$.

If $A \subset B(H)$ is a commutative subalgebra, it is a vector lattice and may be identified with a space $C(\Omega)$ of continuous functions on a locally compact space Ω . A matrix $x \in M_n(A)$ is positive if and only if $[x_{ij}(\omega)] \in M_n^+$ for all $\omega \in \Omega$. Hence we obtain for commutative A (and only for commutative A)

(2.3) $$x \in M_n(A)^+ \Longleftrightarrow x \times \alpha \in A^+ \quad \text{for all} \quad \alpha \in M_n^+ \ .$$

Thus in the commutative case (lattice case) the order structure of $M_n(A)$ is uniquely determined by the order structure of A .

2.2 THE RIESZ SEPARATION PROPERTY IMPLIES THE MATRICIAL RIESZ SEPARATION PROPERTY

An ordered vector space E has the Riesz separation property if the assumption

(2.4) $$x_i, y_i \in E \ , \quad 0, x_i \leq y_j \qquad i,j = 1,\dots,n$$

implies

(2.5) there is a $z \in E^+$ such that $x_i \leq z \leq y_j$, $i,j = 1,\dots,n$

We define diagonal matrices

$$\sigma = \sigma_n = \operatorname{diag}(1,\dots,1,-1,\dots,-1) \in (M_{2n})_h$$
$$a = \operatorname{diag}(y_1,\dots,y_n , -x_1,\dots,-x_n) \in M_{2n}(E)_h$$

and write (2.5) in an artificial form :

(2.6) there is a $z \in E^+$ such that $z \otimes \sigma_n \leq a$.

What happens if a is not diagonal but a general hermitian matrix in $M_{2n}(E)_h = E \otimes_{\mathbb{R}} (M_{2n})_h$ ordered by the rule (2.3)? We are searching for a condition on a which implies the estimation (2.6). If (2.6) is fulfilled then a has the following property:

(2.7) if $\xi \in M_{2n}^{+}$, $\sigma \times \xi \geq 0$ then $a \times \xi \geq 0$.

PROPOSITION 2.1: Let E be a uniformly complete ordered vector space, which has the Riesz separation property. Let $a \in M_{2n}(E)_h$, then condition (2.7) implies the estimation (2.6).

PROOF: We may assume that $E = A(K)$, the space of continuous affine functions on a compact Choquet simplex K . Let

$$g(k) = \inf \{a(k) \times \xi \mid \xi \in M_{2n}^{+}, \quad \sigma \times \xi = 1\}$$

$$f(k) = \sup \{-a(k) \times \eta \mid \eta \in M_{2n}^{+}, \quad \sigma \times \eta = -1\}$$

for $k \in K$. The function g is concave, f is convex and by (2.7) $0,f \leq g$. Denote by $\| \|_1$ the trace norm on the matrices. The set

$$U = \{\alpha \in (M_{2n})_h \mid \text{there exists } \varepsilon > 0 \text{ such that } -\alpha \times \eta - \varepsilon \|\eta\|_1 < \alpha \times \xi + \varepsilon \|\xi\|_1 \text{ for all } \xi,\eta \in M_{2n}^{+}, \ \sigma \times \xi = 1, \ \sigma \times \eta = -1\}$$

is an open convex set in the finite dimensional vector space $(M_{2n})_h$. The functions

$$\gamma(\alpha) = \inf \{\alpha \times \xi \mid \xi \in M_{2n}^{+}, \quad \sigma \times \xi = 1\} ,$$

$$\varphi(\alpha) = \sup \{-\alpha \times \eta \mid \eta \in M_{2n}^{+}, \quad \sigma \times \eta = -1\}$$

are real concave, respectively convex functions on U , hence they are continuous. Thus $g(k) = \gamma(a(k))$ and $f(h) = \varphi(a(k))$ are continuous on K .

By Edward's separation theorem (cf. [1], theorem II, 3.10) there is a $z \in A(K)$ such that $0,f \leq z \leq g$. This implies

$$z(\sigma \times \xi) \leq a(k) \times \xi \qquad \text{f.a. } \xi \in M_{2n}^{+}$$

i.e. $$z \otimes \sigma \leq a$$

in the pointwise order of $M_{2n}(A(K))$.

2.3 B(H) HAS THE MATRICIAL RIESZ SEPARATION PROPERTY

We start with the case where $\dim H = k < \infty$, $B(H) = M_k$ and prove an analogue of proposition 2.1 for matrices.

LEMMA 2.2: Let $a \in M_{2n}(M_k)_h$ and assume that

$$\xi \in M_{2n}(M_k)^+, \quad \sigma_n \times \xi \geq 0 \quad \text{implies} \quad a \times \xi \geq 0 \quad .$$

Then there exists a $z \in M_k{}^+$ such that $z \otimes \sigma_n \leq a$.

PROOF: Let $\eta \in (M_k)_h$, $\eta = \eta^+ - \eta^-$, $\eta^+, \eta^- \in M_k{}^+$ and ε_{ij} the matrix units in M_{2n} . Then we have

$$\xi = \eta^+ \otimes \varepsilon_{n,n} + \eta^- \otimes \varepsilon_{2n,2n} \in M_{2n}(M_k)^+$$

and $\sigma_n \times \xi = \eta$. We identify $M_{2n}(M_k) = M_{2nk}$ in the natural way. The functional

$$\vartheta(\eta) = \inf \left\{ a \times_{2nk} \xi \mid \sigma_n \times_{2n} \xi \geq \eta, \quad \xi \in M_{2n}(M_k)^+ \right\}$$

is well defined, subadditive and positive homogeneous on $(M_k)_h$ By assumption we have $\vartheta(0) \geq 0$ und hence

$$0 = \vartheta(0) \leq \vartheta(\eta) + \vartheta(-\eta) \quad .$$

ϑ is a real valued sublinear functional. By the usual Hahn-Banach theorem there is a linear form φ on $(M_k)_h$ such that

$$\varphi(\eta) \leq \vartheta(\eta) \qquad \text{for all} \quad \eta \in (M_k)_h$$

and there exists a unique $z \in (M_k)_h$ such that

$$\varphi(\eta) = z \times_k \eta \quad .$$

Since $\vartheta(\eta) \leq 0$ for all $\eta \leq 0$ we have $z \geq 0$.
From

$$(z \otimes \sigma_n) \times_{2nk} \xi = z \times_k (\sigma_n \times_{2n} \xi) =$$
$$\varphi(\sigma_n \times_{2n} \xi) \leq \vartheta(\sigma_n \times_{2n} \xi) \leq a \times_{2nk} \xi ,$$

for all $\xi \in M_{2n}(M_k)^+$, we obtain

$$z \otimes \sigma_n \leq a \quad .$$

Now let H be an infinite dimensional Hilbert space. The family of finite dimensional orthogonal projections p is directed upwards and converges to the identity. Moreover $p\,x\,p$ converges in the weak operator topology to $x \in B(H)$. The map $x \longmapsto p\,x\,p$ is completely positive. Using this approximation by finite dimensional operators, the result of Lemma 2.2 and the weak compactness of the unit ball of $B(H)$ we obtain the following theorem:

THEOREM 2.3: Let $a \in M_{2n}(B(H))$ and assume that

(i) $k \in \mathbb{N}$, $\xi \in M_{2n}(M_k)^+$, $\sigma_n \times \xi \geq 0$ implies $a \times \xi \geq 0$.

Then

(ii) there exists a $z \in B(H)^+$ such that $z \otimes \sigma_n \leq a$.

We call the property (i) $\Longrightarrow$(ii) of Theorem 2.3 the <u>matricial Riesz separation property</u>.

In this form the matricial Riesz separation property is given by L. Schmitt [18]. It characterizes the injective von Neumann algebras. In [27] the author gave a similar characteristic property:

THEOREM 2.4: Let $a \in M_{2n}(B(H))$ and assume that

(i') $k \in \mathbb{N}$, $\xi \in M_{2n}(M_k)^+$, $\sigma_n \times \xi = 0$ implies $a \times \xi \geq 0$.

Then

(ii') there exists a $z \in B(H))_h$ such that $z \otimes \sigma_n \leq a$.

For the proof one changes the inequality sign to an equality sign in the definition of ϑ in Lemma 2.2. It does not seem to be obvious that both properties are equivalent.

2.4 THE MATRICIAL RIESZ SEPARATION PROPERTY IMPLIES ARVESON'S EXTENSION THEOREM

THEOREM 2.5: (Arveson [3]) Let A be a C*-algebra (or a matrix ordered space), $E \subset A$ a cofinal *-invariant subspace and $T : E \longrightarrow B(H)$ a completely positive linear map. Then there exists a completely positive extension $\widetilde{T} : A \longrightarrow B(H)$.

PROOF: Let $x \in A^+ \setminus E$. We are looking for a $z \in B(H)^+$ such that

(2.8) $x \otimes \alpha \leq y$ implies $z \otimes \alpha \leq T_m(y)$

for all $m \in \mathbb{N}$, $y \in M_m(E)_h$, $\alpha \in (M_m)_h$. Then we define a completely positive extension $\widetilde{T}$ of T by

$$\widetilde{T}(y + \lambda x) = T(y) + \lambda z \qquad \text{for all } \lambda \in \mathbb{C},\ y \in E .$$

It is sufficient to solve (2.8) for the special matrices

$\alpha = \sigma_n = \text{diag}\,(1,\ldots,1;\ -1,\ldots,-1) \in (M_{2n})_h$, $m = 2n$, $n \in \mathbb{N}$.

The general case follows easily by a transformation $\alpha = \gamma^* \sigma_n \gamma$

with an appropriate rectangular matrix γ .

Let $\xi \in M_k(M_{2n})^+$, $\sigma_n \times \xi \geq 0$, then

$$0 \leq x \otimes (\sigma_n \times \xi) \leq y \times \xi \Longrightarrow 0 \leq T_{2n}(y) \times \xi \quad .$$

By the matricial Riesz separation property there is a $z \in B(H)^+$ such that

$$z \otimes \sigma_n \leq T_{2n}(y) \quad .$$

Let $n_\varkappa \in \mathbb{N}$, $y_\varkappa \in M_{2n_\varkappa}(E)$, $\varkappa = 1,\dots,k$ such that

$$x \otimes (\sigma_{n_1} \oplus \dots \oplus \sigma_{n_k}) \leq y_1 \oplus \dots \oplus y_k \quad .$$

Since $\sigma_{n_1} \oplus \dots \oplus \sigma_{n_k}$ is similar to σ_m , $m = n_1 + \dots + n_k$, there is a $z \in B(H)^+$ such that

$$z \otimes (\sigma_{n_1} \oplus \dots \oplus \sigma_{n_k}) \leq T_{2m}(y_1 \oplus \dots \oplus y_k)$$

i.e.

$$z \otimes \sigma_{n_\varkappa} \leq T_{2n_\varkappa}(y_\varkappa) \quad \text{for} \quad \varkappa = 1,\dots,k \quad .$$

Since bounded closed sets in $B(H)$ are compact in the weak operator topology, there is a $z \in B(H)^+$ such that

$$z \otimes \sigma_n \leq T_{2n}(y) \qquad \text{for all} \quad n \in \mathbb{N}, \quad y \in M_{2n}(E)_h \quad .$$

The proof of the extension theorem is finished as usual by transfinite induction or by Zorn's lemma.

2.5 THE MATRICIAL HAHN-BANACH THEOREM

The usual Hahn-Banach theorem (Banach [4] Chp.II, Thm.1) says: given a sublinear functional $p : E \longrightarrow \mathbb{R}$, a subspace $F \subseteq E$, a linear map $f : F \longrightarrow \mathbb{R}$ dominated by $p|_F$ then there exists a linear extension $\tilde{f} : E \longrightarrow \mathbb{R}$ such that $\tilde{f}(x) \leq p(x)$ for all $x \in E$. The same theorem with exactly the same proof holds, if $\mathbb{R}$ is replaced by an order complete vector lattice (cf. [9]). We want to generalize the Hahn-Banach theorem to $B(H)$ in place of $\mathbb{R}$. Usually the sublinear functional p is constructed by taking the infimum of some set-valued functional. B(H) is an antilattice and it is not possible to take infima. Therefore we consider set-valued functionals. We introduce the relation

$K \preccurlyeq L$ if for every $l \in L$ there is a $k \in K$ with $k \leq l$, where K,L are subsets of $B(H)_h$. Let E be a $\mathbb{K}$-vectorspace ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

A set valued functional $\vartheta: E \longrightarrow B(H)_h$ is called sublinear, if it has the following properties:

(i) $\vartheta(x) \neq \emptyset$ for all $x \in E$

(ii) $\vartheta(x_1 + x_2) \preccurlyeq \vartheta(x_1) + \vartheta(x_2)$ for all $x_1, x_2 \in E$

(iii) $0 \preccurlyeq \vartheta(0)$

(iv) $\vartheta(\lambda x) \preccurlyeq \lambda\vartheta(x)$ for all $x \in E$, $\lambda \in \mathbb{R}^+$.

Clearly, if ϑ takes its values in some commutative order complete subalgebra A , we may take infima in A and $x \longmapsto p(x) = \inf \vartheta(x)$ is a well defined sublinear functional into A .

The notion of set valued sublinear functional becomes useful if it is connected with matrix order:

DEFINITION 2.6: Let E be a $\mathbb{K}$-vector space. A family $\vartheta = (\vartheta_n)_{n\in\mathbb{N}}$ of set valued functionals $\vartheta_n : E \otimes_{\mathbb{R}} (M_n)_h \longrightarrow M_n(B(H))_h$ is called a matricial sublinear functional denoted by $\vartheta : E \longrightarrow B(H)_h$ if ϑ_n $(n \in \mathbb{N})$ fulfils condition (i)-(iii) and the matricial homogeneity condition

(v) $\vartheta_m(x \times \xi) \preccurlyeq \vartheta_n(x) \times \xi$

for all $\xi \in M_n(M_m)^+$, $x \in M_n(E)_h$, $n,m \in \mathbb{N}$.

We say a linear map $T : E \longrightarrow B(H)$ is dominated by a matricial sublinear functional $\vartheta : E \longrightarrow B(H)_h$ if each multiplicity map fulfils

$$\operatorname{re} T_n(x) := \frac{1}{2}(T_n(x) + T_n(x)^*) \preccurlyeq \vartheta_n(x), \quad \text{for all } x \in E \otimes_{\mathbb{F}} (M_n)_h .$$

If ϑ and T have their values in a commutative subalgebra A of $B(H)$ then

$$\operatorname{re} T_1(x) \preccurlyeq \vartheta_1(x) \text{ implies } \operatorname{re} T_n(x) \preccurlyeq \vartheta_n(x) \quad \text{for all } n .$$

This can be seen in the following way : Let $x \in E \otimes_{\mathbb{R}} (M_n)_h$, $\xi \in M_n^{\ +}$.

$$\operatorname{re} T_n(x) \times \xi = \operatorname{re} T_1(x \times \xi) \preccurlyeq \vartheta_1(x \times \xi) \preccurlyeq \vartheta_n(x) \times \xi .$$

Since A is a lattice the matrix order of A is given by (2.3) and hence $\operatorname{re} T_n(x) \preccurlyeq \vartheta_n(x)$. Thus in the commutative case the following matricial Hahn-Banach theorem reduces to the usual Hahn-Banach theorem for order complete lattices.

THEOREM 2.7: ([27]) Let E be a $\mathbb{K}$- vector space, $\vartheta : E \longrightarrow B(H)_h$

a matricial sublinear functional, $F \subset E$ a $\mathbb{K}$-linear subspace and $T : E \longrightarrow B(H)$ a $\mathbb{K}$-linear map dominated by ϑ. Then there exists a $\mathbb{K}$-linear extension $\tilde{T}$ dominated by ϑ. The following proof generalizes the well known usual proof of the Hahn-Banach theorem. The proofs given in [27], chp. 2, are different. In [27] and [28] one finds some applications of this theorem.

PROOF: The complex case follows easily from the real case.
Let $x \in E \setminus F$. We are looking for a $z \in B(H)_h$ such that

$$(2.9) \qquad z \otimes \alpha \leq \vartheta_m(y + x \otimes \alpha) - T_m(y)$$

for all $\alpha \in (M_m)_h$, $y \in E \otimes_{\mathbb{F}} (M_m)_h$, $m \in \mathbb{N}$.
Then we define an extension $\tilde{T} : F \oplus \mathbb{R}x \longrightarrow B(H)$ by

$$\tilde{T}(y + \lambda x) = T(y) + \lambda z \quad \text{for all } \lambda \in \mathbb{R}, \quad y \in F .$$

It is sufficient to solve (2.9) for the special matrices $\alpha = \sigma_n = \mathrm{diag}(1,\ldots 1,-1,\ldots,-1) \in (M_{2n})_h$, $m = 2n$, $n \in \mathbb{N}$. Let $\xi \in M_1(M_{2n})^+$, $\sigma_n \times \xi = 0$ then

$$\begin{aligned} 0 &\leq \vartheta_1((y + x \otimes \sigma_n) \times \xi) - T_1(y \times \xi) \\ &\leq (\vartheta_{2n}(y + x \otimes \sigma_n) - T_{2n}(y)) \times \xi . \end{aligned}$$

By the matricial Riesz separation property of $B(H)$, given in Theorem (2.4), there exists for every $w \in \vartheta_{2n}(y + x \otimes \sigma_n) - T_{2n}(y)$ a $z \in B(H)_h$, such that $z \otimes \sigma_n \leq w$.

From property (ii) and (v) of definition 2.6 of a matricial sublinear functional it follows that

$$\vartheta_{n_1 + \ldots + n_k}(x_1 \oplus \ldots \oplus x_k) \leq \vartheta_{n_1}(x_1) \oplus \ldots \oplus \vartheta_{n_k}(x_k)$$

for $n_\varkappa \in \mathbb{N}$, $x_\varkappa \in E \otimes (M_{n_\varkappa})_h$, $\varkappa = 1,\ldots,k$, $k \in \mathbb{N}$

Using the same idea as in our proof of Arveson's extension theorem (2.5) we find a $z \in B(H)_h$ such that

$$z \otimes \sigma_n \leq \vartheta_{2n}(y + x \otimes \sigma_n) - T_{2n}(y)$$

for all $n \in \mathbb{N}$, $y \in E \otimes_{\mathbb{R}} (M_{2n})_h$.

Property (v) of definition 2.6 implies

$$\gamma^* \vartheta_m(x) \gamma \leq \vartheta_m(\gamma^* x \gamma) \leq \gamma^* \vartheta_m(x) \gamma$$

for every invertible $\gamma \in M_m$, $x \in E \otimes_{\mathbb{R}} (M_m)_h$, $m \in \mathbb{N}$.

Let $\sigma_{k,l} = \text{diag}(1,..1,-1,..-1)$ with k positive entries and l negative entries. Since $\sigma_{k,l} \oplus \sigma_{l,k}$ is similar to σ_{k+l} we obtain for $y \in E \otimes_{\mathbb{R}} (M_{k+l})_h$

$$z \otimes (\sigma_{k,l} \oplus \sigma_{l,k}) \prec \vartheta_{2k+2l}(y \oplus y + x \otimes (\sigma_{k,l} \oplus \sigma_{l,k})) - T_{2k+2l}(y \oplus y)$$
$$\prec (\vartheta_{k+l}(y + x \otimes \sigma_{k,l}) - T_{k+l}(y)) \oplus (\vartheta_{k+l}(y + x \otimes \sigma_{l,k}) - T_{k+l}(y)) .$$

Hence

$$z \otimes \sigma_{k,l} \prec \vartheta_{k+l}(y + x \otimes \sigma_{k,l}) - T_{k+l}(y) .$$

Every invertible $\alpha \in (M_n)_h$ is similar to some $\sigma_{k,l}$, where k is the number of positive eigenvalues of α, l the number of negative eigenvalues, $k+l = n$. Hence

$$z \otimes \alpha \prec \vartheta_n(y + x \otimes \alpha) - T_n(y)$$

for all invertible $\alpha \in (M_n)_h$, $y \in E \otimes_{\mathbb{R}} (M_n)_h$, $n \in \mathbb{N}$.
If α is not invertible, $\alpha + \delta 1\!\!1_n$ is invertible for small $\delta > 0$. For $\delta \to 0$ we obtain (2.9).

The proof is finished as usual by transfinite induction or by Zorn's Lemma.

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Functional Analysis: Surveys and Recent Results III
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ERGODIC THEOREMS IN VON NEUMANN ALGEBRAS

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Ergodic theory has been a large body of the mathematical literature for a long time. In the von Neumann algebra setting it was initiated by Kovács and Szűcs ([7], [8]) in the 60's.

To give a better insight into the subject we start with citing some classical results whose analogues in the algebraic set-up are the main components of this talk. As in ergodic theory, let (Ω, S, μ) be a measure space and let $T:\Omega \to \Omega$ be a measure-preserving transformation. For a complex function f of Ω

$$n^{-1}(f+f\circ T+\ldots+f\circ T^{n-1})$$

is denoted by $s_n(f)$. By the statistical ergodic theorem of von Neumann $s_n(f)$ converges in L^2-norm for every $f \in L^2(\mu)$ and Birkhoff's individual ergodic theorem asserts that if $f \in L^1(\mu)$ then $s_n(f)$ converges μ-almost everywhere. In fact, $U_T f = f\,T$ is a unitary in $L^2(\mu)$ and for every contraction U in a Hilbert space $n^{-1}\sum_{i=0}^{n-1} U^i$ converges strongly to a projection P satisfying $PU = UP = P$. The mean ergodic theorem of Alaoglu and Birkhoff concerns a semigroup of contractions. If G is such a semigroup in the Hilbert space H then there exists a projection P in the closure of conv$G$ such that $P\xi$ is the unique G-fixed point in $\overline{\text{conv}G\xi}$ for every $\xi \in H$. This result probably inspired both the basic theorem of Kovács and Szűcs and an abstract mean ergodic theory ([20], III.7.). The Kovács-Szűcs theorem states that if G is a group of automorphisms of a von Neumann algebra A possessing a separating family of G-invariant normal states then the w-closed convex hull of Ga, $a \in A$, contains a unique fixed point $E(a)$ (see Theorem 1 below for the details). It was observed in [10] that this fits very well in the abstract mean ergodic theory. A semigroup G of bounded operators on a Banach space A is given and the closure of conv$G$ contains a projection E such that $E \circ g = g \circ E = E$ for all $g \in G$.

In the first part of this note we deal with mean ergodic theorems in the von Neumann algebra setting and in the second part we turn to "pointwise" ergodic theory. The individual theorem treated covers Birkhoff's theorem in the case $f \in L^\infty$.

A always will denote a von Neumann algebra. We recall that a linear map $\alpha: A \to A$ is called Schwarz map when $\alpha(a)^*\alpha(a) \le \alpha(a^*a)$ for every $a \in A$. Schwarz maps are positive and so bounded. It is known that every 2-positive map is a Schwarz map. The next result is a generalization of the Kovács-Szűcs theorem.

THEOREM 1 ([10]). *If G is a semigroup of w-continuous Schwarz maps on* A *and there is a family Φ of normal states such that*

(i) $\varphi(g(a)) \le \varphi(a)$ $(\varphi \in \Phi,\ g \in G,\ a \in A^+)$,

(ii) if $a \in A^+$ *and* $\varphi(a) = 0$ *for all* $\varphi \in \Phi$ *then* $a = 0$,

then the closure of conv$G$ in the weak operator topology contains a projection E which is a G-invariant conditional expectation onto the fixed point subalgebra of* A .

We show an example how Theorem 1 (or the Kovács-Szűcs theorem) can be applied to approximate conditional expectations by convex combinations of automorphisms. For $B \subset A$ we denote by B^c the relative commutant of B in A . If $B = B^{cc}$ then $G = \{\mathrm{Ad}u : u$ is a unitary in $B^c\}$ is a group of inner automorphisms of A and B is the fixed point algebra of G . Assume that A possesses a faithful unitarily invariant normal state τ (in other words, A is supposed to be finite). So by Theorem 1 there is a net$(\beta_i) \subset$ conv$G$ such that $E(a) = \text{w-lim}\,\beta_i(a)$ where $E: A \to B$ is the τ-preserving expectation.

Let $f: A^+ \to \mathbb{R}$ be a lower w-semicontinuous convex function which is unitarily invariant $(f(u^*au) = f(a)$ if u is a unitary in A).

Then

$$f(E(a)) = f(\text{w-lim}\ \beta_i(a)) \le \lim\inf f(\beta_i(a)) = f(a) .$$

This situation occurs in some physical applications (see [12]). Let $\varphi(.) = \tau(\rho.)$ be a state of A. The entropy of φ is defined as $H(\varphi) = -\tau(\rho\log\rho)$. Since the function $\rho \mapsto -\tau(\rho\log\rho)$ can be put in the role of f above we obtain $H(\varphi|_B) \le H(\varphi)$. A similar argument yields the decrease of the relative entropy of two states under some kind of expectation ([12], [23]).

The following theorem was proved in [11] for cyclic G and a locally compact version is treated in [24].

THEOREM 2. *Let G, Φ, E be as in Theorem 1. If G is a countable commutative group, then*

$$A_o = \{a \in A : E(a) \in \overline{\text{conv}\, Ga}^{\,u}\}$$

is a w-dense subspace of A.

We indicate the proof. Since G is amenable there is a summing sequence (U_n) ([5]), that is, $U_1 \subset U_2 \subset \ldots \subset G$ and

$$\frac{|g+U_n|}{|U_n|} \to 0$$

as $n\to\infty$ for every $g \in G$. Let

$$A_n(a) = |U_n|^{-1} \sum_{g\in U_n} g(a).$$

Then $A_n(a) \overset{w}{\to} E(a)$ for every $a\in A$ (cf.[5],3.3.Thm).
If $f \in \ell^1(G)$ and $\sum_{g\in G} f(g) = 0$ then

$$a_f = \sum_{g\in G} f(g)g(a) \in A_o$$

for every $a \in A$. More specifically, $\|A_n(\alpha_f)\| \to 0$ as $n \to \infty$. So for each $a \in A$ and $m \in \mathbb{N}$ we have

$$a_m = a - A_m(a) + E(a) \in A_o$$

and on the other hand $a_m \overset{w}{\to} a$.

It seems to be a hard question to decide if the subspace A_o contains a w-dense $*$-subalgebra. As far as we know this problem was posed in [11] and it is open even for cyclic G. When G is commutative the answer is affirmative.

From now on we fix a faithful normal state ψ of A. We may assume that A acts on a Hilbert space H and ψ is determined by a cyclic separating vector ξ, i.e. $\psi(a) = \langle a\xi, \xi\rangle$. In such a situation we have the tools of the Tomita-Takesaki theory at our disposal. J is the canonical conjugation on H and $a \mapsto Ja^*J$ is an antiisomorphism of A onto A'. For $a \in A$

$$\gamma_a(.) = \langle .\xi, Ja^*J\rangle$$

is a normal functional of A. So γ is a positivity preserving dense embedding of A into A_* ([6], [16], [22], [21], 1.8.). We define $\| \|_1$ on A as

$$\|a\|_1 = \|\gamma_a\| \quad .$$

($\|.\|_1$ is the analogue of the L^1-norm and it is really that when ψ is a trace.) We recall that for $a \geq 0$ we have $\|a\|_1 = \psi(a)$ and if $b \in A^{sa}$ then

$$\|b\|_1 = \inf\{\psi(a_1)+\psi(a_2) : a_1, a_2 \geq 0 , \quad b = a_1 - a_2\} \quad .$$

When $\alpha: A \to A$ is positive and satisfies $\psi \circ \alpha \leq \psi$ then it is quite straightforward that $\|\alpha(a)\|_1 \leq \|a\|_1$. Hence the next theorem generalizes Theorem 1.

THEOREM 3. *Let* A *be a von Neumann algebra with a faithful normal state* ψ. *If* G *is a semigroup of normal selfmaps of* A *such that* *(i)* $\| g(a) \| \leq \| a \|$ $(a \in A\ ,\ g \in G\ )$

(ii) $\| g(a) \|_1 \leq \| a \|_1$ $(a \in A\ ,\ g \in G\ )$

then G *is mean ergodic. (In other words, the closure of* conv$G$ *contains a zero.)*

The proof is based on the classical Alaoglu-Birkhoff theorem ([1]) and the Calderon-Lions interpolation theorem ([19], IX.20.). We use the fact that the spatial L^2 space with respect to ψ is an interpolation space between A and A_* (see [22] and its references). By assumption every $g \in G$ is a contraction with respect both to $\| . \|$ and to $\| . \|_1$. According to the interpolation theorem they are norm decreasing in $L^2(\psi)$ and admit an extension to a contraction. However, $L^2(\psi)$ is isomorphic to H ([22], Theorem 23) and the relations between the topology of H and the operator topologies of A are well-known ([3],I.Chap.4.Prop 4 or [21], 2.12.).

As a corollary we state the adjoint of Theorem 3 for a single $\alpha : A \to A$. It can be proved directly without results of interpolation type ([16], Theorem 14).

COROLLARY. *If* α_* *is a contraction of* A_* *such that* $\| \alpha_*(\varphi) \|_\infty \leq \| \varphi \|_\infty$ *for every* $\varphi \in A_*$ *then*

$$S_n(\varphi) = n^{-1} \sum_{i=0}^{n-1} \alpha_*^i (\varphi)$$

converges in norm as $n \to \infty$ *for each* $\varphi \in A_*$.

Here $\| \varphi \|_\infty$ is defined as follows:

$$\| \varphi \|_\infty = \sup\{ |\varphi(a)| \ :\ a \in A\ ,\ \| a \|_1 \leq 1 \} .$$

Equivalently, $\| \varphi \|_\infty = \| a \|$ if there is $a \in A$ with $\varphi = \gamma_a$, otherwise $\| \varphi \|_\infty = \infty$ ([16]) .

Now we leave mean ergodic theorems and turning to pointwise convergence in the rest of the paper we assume that $G = \{\alpha^n : n \in \mathbb{N}\}$. By the results above we have

$$S_n(a) = n^{-1} \sum_{i=0}^{n-1} \alpha^i(a) \overset{w}{\to} E(a)$$

for every $a \in A$ and we know about plenty of elements $b \in A$ such that $S_n(b) \to E(b)$ in norm (Theorem 2). This fact and the measure theoretic ergodic theory make us guess that the convergence may remain true with respect to some topology between the w-convergence and the norm topology. In fact, it is a pleasure to show that $S_n(a) \to E(a)$ strongly and Lance ([11]) imitated the almost everywhere convergence of functions and obtained the following result (see also [2], [9], [14]) .

THEOREM 4. *If* $\alpha : A \to A$ *is a Schwarz map such that* $\psi \alpha \le \psi$ *then for every* $a \in A$ *and* $\varepsilon > 0$ *there is a projection* $p \in A$ *such that* $\psi(1-p) < \varepsilon$ *and*

$$\|(S_n(a) - E(a))p\| \to 0 \quad \text{as} \quad n \to \infty \quad .$$

Shortly, we can say that $S_n(a) \to E(a)$ φ-almost uniformly. Lance's result is a perfect individual ergodic theorem in the von Neumann algebra context. A curiosity of the φ-almost uniform convergence is that its additivity is not clear. To avoid this slightly awkward feature we introduce the quasi-uniform convergence.

Let $(a_n) \subset A$ be a bounded sequence. $a_n \to a$ quasi-uniformly if for every projection $0 \ne p \le A$ there is a non-zero projection $A \ni q \le p$ such that $\|(a_n - a)q\| \to 0$. Quasi-uniform convergence is stronger than the almost uniform one and its additivity is trivial due to the following simple lemma.

LEMMA. *Assume that* $a_n \to a$ *quasi-uniformly and let* ψ *be a normal faithful state. Then for every* $\varepsilon > 0$ *and for every projection* $p \in A$ *there exists a projection* $q \le p$ *such that* $\psi(p-q) < \varepsilon$ *and* $\|(a_n - a)q\| \to 0$ *as* $n \to \infty$.

THEOREM 5 ([15]). *Under the conditions of Theorem 4*

$$S_n(a) \to E(a)$$

quasi-uniformly as $n \to \infty$ *for every* $a \in A$.

Let B be a w-dense norm separable sub-C*-algebra of A and assume that ψ is tracial. Radin proved ([18]) that there is a sequence $p_n \nearrow 1$ of projections in A such that

$$\| [S_n(a) - E(a)] p_m \| \to 0$$

as $n \to \infty$ for any $m \in \mathbb{N}$ and $a \in B$. He posed the question if the assumption on ψ can be relaxed. As an easy consequence of Theorem 5, Radin's result is true for every ψ .

Some other theorems can also be generalized by means of the quasi-uniform convergence (see [15]) .

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Note added in proof:

After the completion of this contribution B. Kümmerer called my attention to his joint paper with U. Groh (Math.Scand., 50 (1982), 269-285) in which Theorem 3 is proved.

Functional Analysis: Surveys and Recent Results III
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AUTOMATIC CONTINUITY OF HOMOMORPHISMS FROM C*-ALGEBRAS

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Let A and B be Banach algebras, and let $\theta: A \to B$ be a homomorphism. The basic automatic continuity problem is to give algebraic conditions on A and/or B which ensure that θ is automatically continuous. In this lecture, I wish to describe some of the known results and some of the problems which remain open, concentrating on the special case in which A is a C*-algebra.

Let me first suggest one reason why questions concerning the possible continuity of homomorphisms are of significance in Banach algebra theory. It is not, of course, the case that one would use results in this area to show that an explicitly specified homomorphism is continuous. A Banach algebra has both an algebraic and a topological structure, and these structures are joined together, apparently loosely, by the requirement that the algebraic operations be continuous. To resolve a problem in automatic continuity theory one must look more closely at these two structures, and in particular one must study the relationships between them. It seems that this study has led to several striking insights into the structure of Banach algebras: both the positive results obtained and the counter-examples that have been constructed have played a rôle in this, and I believe that when the open problems are resolved, we will obtain similar illumination. I hope that the results which are referred to below will help to substantiate these claims.

Automatic continuity theory is already a well-surveyed area. A number of results are given in the standard texts: see [29] and [7], for example. The theory of the automatic continuity of linear operators between Banach spaces was discussed in 1976 in Allan Sinclair's book ([32]). An important development of this theory is the work of Ernst Albrecht and Michael Neumann on the automatic continuity of certain linear operators between spaces of functions and distributions: for a survey of this work, see the lecture [27] given at an earlier conference in this series. The survey [9] , which appeared in 1978, concentrates on the theory of homomorphisms between Banach algebras and of derivations into Banach bimodules, and the article [25] of Kjeld Laursen discusses the automatic continuity of "intertwining operators", a class of linear operators which includes both homomorphisms and derivations, concentrating on the automatic continuity of intertwining operators from Banach algebras of differentiable functions. Finally, let me mention the volume [5] , the proceedings of a conference held in Long Beach in 1981, where many relevant articles - both surveys and new results - can be found. A number of open questions can be found at the end of this volume.

I wish to thank the organisers of this conference for their kind invitation to attend, and for arranging that this paper be typed in Paderborn, and to thank Ms. B. Duddeck for her excellent typing.

1. EARLY RESULTS

We first introduce some standard notation. Let A be a Banach algebra. A character on A is a non-zero homomorphism $A \to \mathbb{C}$. Each character on A is automatically continuous. We write Φ_A for the character space of A, and we write rad A for the radical of A. A basic notion in automatic continuity theory is that of the separating space: if $\theta: A \to B$ is a homomorphism, then the separating space of θ is the set

$$\mathcal{S}(\theta) = \{b \in B : \textit{there exists a sequence } (a_n) \subset A \textit{ with } a_n \to 0 \textit{ and } \theta a_n \to b\}.$$

Clearly, by the closed graph theorem, θ is continuous if and only if $\mathcal{S}(\theta) = \{0\}$. The space $\mathcal{S}(\theta)$ is a closed linear subspace of B, and it is a bi-ideal if $\theta(A)$ is dense in B. Basic properties of $\mathcal{S}(\theta)$ are given in [32, § 1].

Automatic continuity theory could not have had an earlier beginning in the theory of Banach algebras because the first result appears in the seminal work of Gelfand in 1941 ([16]) (although Gelfand does not use the notion of the separating space). Let $\theta: A \to B$ be a homomorphism, and let $b \in \mathcal{S}(\theta)$, say $b = \lim \theta(a_n)$, where $a_n \to 0$. Take $\varphi \in \Phi_B$. Then $\varphi \circ \theta \in \Phi_A$, and so $\varphi(b) = \lim \varphi(\theta(a_n)) = 0$. Thus, it is always true that

$$\mathcal{S}(\theta) \subset \cap\{\ker \varphi : \varphi \in \Phi_B\}. \qquad (1)$$

In general, this tells us little, but, if B is commutative, it shows that each element of $\mathcal{S}(\theta)$ is a quasi-nilpotent in B: we shall note below that it is an open problem whether or not this holds for a general (non-commutative) Banach algebra B. If B is commutative and semi-simple, then (1) implies that $\mathcal{S}(\theta) = \{0\}$. Thus, we have the following result, essentially given in [16, Satz 17].

1.1 <u>Theorem.</u> *Let* $\theta: A \to B$ *be a homomorphism. If* B *is commutative and semi-simple, then* θ *is automatically continuous.*

The condition in Theorem 1.1 is imposed on the range algebra B. In this article, our main interest is the case when the domain algebra A is supposed to be a C*-algebra. We shall consider both commutative and non-commutative C*-algebras, beginning with commutative algebras.

Of course, each unital, commutative C*-algebra is isometrically isomorphic to a Banach algebra $C(X)$, the set of all complex-valued, continuous functions on a compact Hausdorff space X, where the norm is the uniform norm $|\cdot|_X$.

The first result about homomorphisms from $C(X)$ was given by Kaplansky in 1949 ([23]; see [29, 3.7.7], [32, 10.1]).

1.2 <u>Theorem.</u> *Let* $\theta: C(X) \to B$ *be a monomorphism. Then* $\|\theta(f)\| \geqslant |f|_X \; (f \in C(X))$.

Thus, there is an incomplete algebra norm on $C(X)$ if and only if there is a discontinuous monomorphism from $C(X)$ into a Banach algebra.

The first substantial study of homomorphisms from the algebras $C(X)$ is due to Bade and Curtis in 1960 ([6]). Before stating their main result, we give some standard notation.

Let X be a compact space, and let $x \in X$. Then

$$M_x = \{f \in C(X) : f(x) = 0\},$$
$$J_x = \{f \in C(X) : f^{-1}(0) \text{ is a neighbourhood of } x\}.$$

Thus, M_x is a maximal ideal of $C(X)$, and $J_x \subset \bar{J}_x = M_x$. A <u>radical homomorphism</u> is a non-zero homomorphism ν from a maximal ideal M_x of $C(X)$ such that $\overline{\nu(M_x)}$ is a radical Banach algebra. It is easy to see that, if ν is a radical homomorphism, then $\nu | J_x = 0$, and so ν is discontinuous.

1.3 <u>Theorem.</u> *Let* θ *be a homomorphism from* $C(X)$ *into a Banach algebra* B. *Then either* θ *is continuous, or there is a non-empty, finite subset* $\{x_1, \ldots, x_n\}$ *of* X, *a continuous homomorphism* $\mu: C(X) \to B$, *and a linear map* $\nu: C(X) \to B$ *such that* $\theta = \mu + \nu$ *and* $\nu = \nu_1 + \cdots + \nu_n$, *where* $\nu_i | M_{x_i}$ *is a radical homomorphism.*

Thus, there is a discontinuous homomorphism from $C(X)$ into a Banach algebra if and only if there is a radical homomorphism from a maximal ideal of $C(X)$. The set $\{x_1,\dots,x_1\}$ is the <u>singularity set</u> of the homomorphism θ.

The above theorem is proved in [32, 10.3] and is discussed in [9, 9.1].

There is an extension of Theorem 1.3 that will be required later. We write βY for the Stone-Čech compactification of a completely regular topological space Y.

Let θ be a homomorphism from $C(X)$, as in Theorem 1.3. Suppose that θ is discontinuous, and let F be the singularity set of θ. For each $f \in C(X)$, let $\hat{f}$ be the continuous extension of $f|(X\setminus F)$ to $\beta(X\setminus F)$. By applying essentially the argument of Bade and Curtis to the algebra $\{\hat{f} : f \in C(X)\}$ on $\beta(X\setminus F)$, Johnson ([22]) showed that there is a non-empty, finite subset E of $\beta(X\setminus F)\setminus(X\setminus F)$ with properties analogous to those of the original singularity set F. In particular, we have the following result. We write $C_0(X\setminus F)$ for $\{f \in C(X) : f|F = 0\}$, and, for $p \in \beta(X\setminus F)$, set

$$J_p^\beta = \{f \in C_0(X\setminus F) : \hat{f}^{-1}(0) \textit{ is a neighbourhood of } p \textit{ in } \beta(X\setminus F)\}.$$

1.4 <u>Theorem.</u> *Let* $\theta: C(X) \to B$ *be a discontinuous homomorphism, and let* F *be the singularity set of* θ. *Then there is a non-empty, finite subset* $\{p_1,\dots,p_m\}$ *of* $\beta(X\setminus F)\setminus(X\setminus F)$, *a continuous homomorphism* $\mu: C(X) \to B$, *and a linear map* $\nu: C(X) \to B$ *such that* $\theta = \mu + \nu$ *and such that* $\nu = \nu_1+\dots+\nu_m$, *where* $\nu_i|C_0(X\setminus F)$ *is a non-zero homomorphism with* $\ker \nu_i \supset J_{p_i}^\beta$.

Thus, there is a discontinuous homomorphism from $C(X)$ if and only if there is a finite subset F of X and a non-zero homomorphism from $C_0(X\setminus F)/J_p^\beta$ for some $p \in \beta(X\setminus F)\setminus(X\setminus F)$.

2. <u>POSITIVE RESULTS FOR NON-COMMUTATIVE ALGEBRAS</u>

Let $(A, \|.\|)$ be a Banach algebra. Then A has a <u>unique complete norm</u> if each norm with respect to which A is a Banach algebra is equivalent to the given norm $\|.\|$. Thus, if A has a unique complete norm, the algebraic structure of A determines the topological structure of A. It is an immediate consequence of Theorem 1.1 that each commutative, semi-simple Banach algebra does have a unique complete norm. The outstanding question that remained open when the treatise of Rickart ([29]) was written in 1960 was whether or not each semi-simple Banach algebra has a unique complete norm.

This problem was discussed by Rickart (see [29, II, § 5]) in terms of the separating space, and a number of partial results were obtained.

The problem was finally resolved, positively, by B. E. Johnson in 1967 ([19]), and a number of proofs are now available (see [7, 25.9] , [32, 6.12], and, for an extension of the result, [18]). Here, I wish to draw attention to a remarkable new proof of the result due to Aupetit ([4]). Write $\nu(a)$ for the spectral radius of an element a of a Banach algebra A. Aupetit's proof uses the fact that $\nu \circ f$ is subharmonic on U whenever $f : U \to A$ is analytic. The proof does not use any theory of representations, and so is an "internal" proof. This is the exact form of the result:

2.1 Theorem. *Let* A *and* B *be Banach algebras, and let* $\theta: A \to B$ *be a homomorphism. Then*

$$\nu(\theta a) \leq \nu(b + \theta a) \qquad (a \in A, \quad b \in \mathcal{S}(\theta)). \tag{2}$$

Suppose further that θ is an epimorphism. Then, for each $b \in \mathcal{S}(\theta)$, we can take $a \in A$ with $\theta a = -b$, and so $\nu(b) = 0$. Thus, $\mathcal{S}(\theta)$ is contained in the set $\mathcal{Q}(B)$ of quasi-nilpotent elements of B, and we know that $\mathcal{S}(\theta)$ is a closed bi-ideal in B. Hence, we have the following corollary.

2.2 Corollary. *Let* $\theta: A \to B$ *be an epimorphism. Then*

$$\mathcal{S}(\theta) \subset \text{rad } B. \tag{3}$$

If B *is semi-simple, then* θ *is continuous, and in particular a semi-simple Banach algebra has a unique complete norm.*

I am supposed to be talking about C*-algebras. Let us now turn to that case. There are basically two methods for proving that homomorphisms from C*-algebras are automatically continuous.

To describe the first method, due to Johnson ([21]), we introduce a second set which plays an important role in automatic continuity theory. Let $\theta: A \to B$ be a homomorphism with separating space $\mathcal{S}(\theta)$. Then

$$I(\theta) = \{a \in A : \theta(a)\mathcal{S}(\theta) = \mathcal{S}(\theta)\theta(a) = \{0\}\}.$$

The set $I(\theta)$ is the continuity ideal of θ. Clearly, $I(\theta)$ is a bi-ideal in A, and it is rather straightforward to check ([32, 1.3]) that

$$I(\theta) = \{a \in A : \text{the maps } x \mapsto \theta(ax),\ x \mapsto \theta(xa),\ A \to B, \text{ are both continuous}\}.$$

For example, let $\theta: C(X) \to B$ be a homomorphism with singularity set $\{x_1,\ldots,x_n\}$. Then, by 1.3, $I(\theta) \supset J_{x_1} \cap \cdots \cap J_{x_n}$, and so $\overline{I(\theta)} = M_{x_1} \cap \cdots \cap M_{x_n}$, a closed ideal of finite codimension in $C(X)$.

2.3 Theorem. *Let* $\theta: A \to B$ *be a homomorphism from a* C*-*algebra* A *into a Banach algebra. Then:*

(i) $\overline{I(\theta)}$ *has finite codimension in* A;

(ii) θ *is continuous if and only if* $I(\theta)$ *is closed.*

Proof. *(i)* Let C be a commutative C*-subalgebra of A. Then $C \cap \overline{I(\theta)}$ has finite codimension in C by the above remark, and this is sufficient to imply *(i)* (see [32, 12.1]).

(ii) If θ is continuous, then $I(\theta) = A$. Now suppose that $I(\theta)$ is closed, and let $a_n \to 0$ in $I(\theta)$. Since each closed ideal in a C*-algebra has a bounded approximate identity, there exists $b, c_n \in I(\theta)$ with $a_n = bc_n$ and $c_n \to 0$ ([7, 11.12]), and so $\theta(a_n) = \theta(bc_n) \to 0$. Thus, θ is continuous on $I(\theta)$, and so, by *(i)*, θ is continuous on A.

2.4 Corollary. *Let* A *be a unital* C*-*algebra with no proper, closed bi-ideals which have finite codimension. Then each homomorphism from* A *is continuous.*

Proof. Necessarily, $\overline{I(\theta)} = A$, and so $I(\theta) = A$ because A is unital.

Let H be a Hilbert space. The corollary shows that each homomorphism from the

C*-algebra $B(H)$ is continuous. However, it does not apply to $K(H)$, the algebra of compact linear operators on H, because $K(H)$ is not unital. There is a device to circumvent the difficulty in this case: it is also taken from [21].

Let $A \subset B(H)$, and let $M(A)$ be the <u>multiplier algebra</u> of A (see [28, § 3.12]). Set

$$I_M(\theta) = \{m \in M(A) : \theta(am)S(\theta) = S(\theta)\theta(ma) = \{0\} \quad (a \in A)\}.$$

Then, essentially as before, it can be shown that $\overline{I_M(\theta)}$ has finite codimension in $M(A)$.

In the special case that $A = K(H)$, we have $M(A) = B(H)$, so that $I_M(\theta) = B(H)$, and again θ is continuous.

The best version of Theorem 1.3 for general C*-algebras is due to Laursen and Sinclair ([31], [26]).

2.5 <u>Theorem.</u> *Let* $\theta: A \to B$ *be a homomorphism from a* C*-*algebra* A. *Then there is a finite-dimensional subspace* F *of* A *such that* $F \oplus \overline{I(\theta)} = A$ *and* $F + I(\theta)$ *is a dense subalgebra of* A. *Further,* $\theta = \mu + \nu$, *where* $\mu: A \to B$ *is a continuous homomorphism which coincides with* θ *on* $F + I(\theta)$, *and* $\nu: A \to B$ *is a linear map such that* $\nu|\overline{I(\theta)} : \overline{I(\theta)} \to S(\theta)$ *is a homomorphism.*

The second method is applicable to C*-algebras which have many projections. Let me give the basic idea. We start with the <u>main boundedness theorem</u> (first given in the commutative case in [6], and in the non-commutative case in [8]).

2.6 <u>Theorem.</u> *Let* $\theta: A \to B$ *be a homomorphism between Banach algebras. Suppose that there exist sequences* (a_n), (b_n) *in* A *such that (i)* $b_m b_n = 0$ $(m \neq n)$, *and (ii)* $a_n b_n = a_n$ $(n \in \mathbb{N})$. *Then* $\sup\{\|\theta(a_n)\|/\|a_n\|\,\|b_n\|\} < \infty$.

A <u>projection</u> in a C*-algebra A is an element p such that $p = p^2 = p^*$. Two projections $p,q \in A$ are <u>orthogonal</u> if $pq = 0$, and they are <u>equivalent</u> if there exists $u \in A$ with $uu^* = p$ and $u^*u = q$. In this case,

$$p = p^2 = uu^*uu^* = uqu^*. \qquad (4)$$

Let $\theta: A \to B$ be a homomorphism from a unital C*-algebra A. The special hypothesis we make on A is the following:

each projection in A *is the sum of two equivalent, orthogonal projections.* (*)

Suppose that (*) holds, and that 1 is the identity of A. Then there exist two sequences (p_n) and (q_n) of projections in A such that, for $n \in \mathbb{N}$:

(i) $1 = p_1 + q_1$, $p_n = p_{n+1} + q_{n+1}$;

(ii) $p_n q_n = 0$;

(iii) p_n is equivalent to q_n.

Clearly, $q_m q_n = 0$ $(m \neq n)$.

Let $I(\theta)$ be the continuity ideal of θ. We <u>claim</u> that $q_k \in I(\theta)$ for some k. For if $q_n \notin I(\theta)$ for each n, there exists $x_n \in A$ with $||x_n|| = 1$ and $||\theta(x_n q_n)|| > n\,||q_n||^2$. Apply 2.6 with $a_n = x_n q_n$, $b_n = q_n$. Then, by that theorem,

$$\sup\{||\theta(x_n q_n)||/||x_n q_n||\ ||q_n||\} < \infty.$$

But $||\theta(x_n q_n)||/||x_n q_n||\ ||q_n|| > n$ by construction, a contradiction. Thus, the claim holds.

Suppose that $q_k \in I(\theta)$. Since p_k is equivalent to q_k, it follows from (4) that $p_k \in I(\theta)$. Thus, $p_{k-1} = p_k + q_k \in I(\theta)$, and eventually we see that $1 \in I(\theta)$. But this says that θ is continuous.

Thus, if (*) holds, then each homomorphism from the C*-algebra A is automatically continuous. Hypothesis (*) holds, for example, in each von Neumann algebra which has no direct summand of Type I ([33, V.I.35]).

3. <u>DISCONTINUOUS HOMOMORPHISMS FROM C(X)</u>

Throughout this section, we take X to be a compact Hausdorff space, and we return to the question of the existence of discontinuous homomorphisms from C(X). We know that such a homomorphism exists if and only if there is a radical homomorphism from a maximal ideal of C(X).

Such a homomorphism was constructed for each infinite set X in [10] and, independently, in [12], and the two constructions are described in [9, § 9]. This description will not be repeated here, but I should like to mention a few points.

Let $C_{\mathbb{R}}(X)$ denote the set of real-valued functions on X, and set $f \prec g$ if $f(x) \prec g(x)$ $(x \in X)$. Then $\prec$ is a partial order on $C_{\mathbb{R}}(X)$, and we obtain an induced partial order on the quotient algebra M_x/J_x for each $x \in X$. In general, this partial order may be very complicated. But consider the special case in which $X = \beta\mathbb{N}$, and take $p \in \beta\mathbb{N} \setminus \mathbb{N}$. Then the induced order is a <u>total</u> order on M_p/J_p. Also, J_p is a prime ideal in $C_{\mathbb{R}}(\beta\mathbb{N})$. Let K_p be the quotient field of the integral domain M_p/J_p. Then K_p is also a totally ordered set, and K_p has the following properties:

(i) K_p is a real-closed field (i.e., $K_p(\sqrt{-1})$ is algebraically closed);

(ii) K_p is an η_1-set (i.e., for each pair (S,T) of countable subsets of K_p with $S < T$, there exists $a \in K_p$ with $S < \{a\} < T$);

(iii) K_p has cardinality $2^{\aleph_0}$.

The main part of [10] is the construction of an algebra monomorphism from the set of finite elements of K_p into a radical Banach algebra. The proof that such a monomorphism exists uses the order structure of K_p, and it also requires the assumption of the continuum hypothesis (CH): the proof involves transfinite induction, and one must well-order a certain subset of K_p which has cardinality $2^{\aleph_0}$ in such a way that each element has only countably many predecessors.

It can be seen that, in the above formulation, the problem is very algebraic. Its resolution has a number of interesting algebraic applications which seem distant from the original discussion of homomorphisms from $C(X)$.

Here is an important ingredient of the proof. It is proved by Johnson in [22] that any two real-closed, totally ordered η_1-fields of cardinality $\aleph_1$ are in fact isomorphic as real algebras. Thus, in particular, if CH holds, then any two fields K_p are isomorphic. However, if CH does not hold, then different K_p's may have very different structures.

The first example of an embedding of an interesting algebra into a radical

Banach algebra is due to G.R. Allan in 1972 ([2]). Let F be the algebra of all formal power series in one variable, let R be a radical Banach algebra, and let $R^{\#}$ denote the Banach algebra formed by adjoining an identity to R. Then Allan proved that there is an embedding of F in $R^{\#}$ if and only if there is a non-zero element a of R such that $a \in \overline{a^2R}$, and he gave examples of algebras R which satisfy this condition.

There is a sense in which Esterle's example of a discontinuous homomorphism from C(X) involves the construction of an embedding from an algebra of formal power series in infinitely many variables into a radical Banach algebra: see the description in [9, § 9].

In the original papers, examples are given of radical Banach algebras R such that there is a discontinuous homomorphism $C(X) \to R^{\#}$. For example, it is shown in [12] that this is true for each radical Banach algebra with a bounded approximate identity. However, an example of a radical algebra without a bounded approximate identity for which the result holds is given in [10]. Thus, the problem arises of characterizing the radical Banach algebras R such that there is a discontinuous homomorphism $C(X) \to R^{\#}$ for each infinite X.

The solution of this problem is due to Esterle. There are two particular points of interest. First, the apparently rough condition on R of Allan, mentioned above, which was known to be necessary, is proved to be sufficient. Second, it was discovered that the existence of certain semi-groups in R is relevant: This observation led Esterle to a remarkable classification of radical Banach algebras in terms of the semi-groups which they contain ([15]). Let A be a Banach algebra, and let S be a semi-group in $\mathbb{C}$. Then A contains a <u>semi-group over S</u> if there exists a non-zero map $\psi: S \to A$ such that $\psi(s_1)\psi(s_2) = \psi(s_1s_2)$ $(s_1,s_2 \in S)$. The semi-group is <u>continuous</u> if ψ is continuous.

3.1 <u>Theorem.</u> (CH). *Let* R *be a commutative radical Banach algebra. Then the following conditions on* R *are equivalent:*

(i) *there is a non-zero element* $a \in R$ *with* $a \in \overline{a^2R}$;

(ii) *there is an embedding* $F \to R^{\#}$;

(iii) *there is a semi-group over* $\mathbb{Q}_0^+$ *in* R;

(iv) *there is a discontinuous homomorphism* $C(X) \to R^{\#}$ *for each infinite* X;

(v) for each commutative integral domain A *without identity and of cardinality* $2^{\aleph_0}$, *there is an embedding of* A *in* R.

Here, $\mathbb{Q}^+_o$ denotes the set of strictly positive rationals. The equivalence of *(i)* and *(ii)*, and the fact that *(iv)* implies *(i)* is due to Allan ([2]). The equivalence of *(iv)* and *(v)* is in [13], that *(i)* implies *(iii)* and that *(iii)* implies *(iv)* is in [14], and a direct proof that *(iii)* implies *(i)* is [15, Theorem 3.3].

There is one further condition which is conceivably equivalent to the above. It is:

(vi) there is a continuous semi-group over $\mathbb{R}^+_o$ *in* R.

Certainly, *(vi)* implies *(iii)*. It is unlikely that *(iii)* implies *(vi)*, and the radical Banach algebra $\ell^1(\mathbb{Q}^+_o, e^{-t^2})$ is probably a counter-example, but this has not been proved (see [15, Question 6]). If *(vi)* were equivalent to *(iii)*, the proof of Theorem 3.1 could be simplified.

The proof of Theorem 3.1 can surely never be short, but we now have a proof that is considerably shorter, more comprehensive, and more incisive, than the original proofs. This proof will be presented in [11]. Two features of the new proof are the use of elementary valuation theory to classify and unify a number of algebraic constructions, and the use of the Mittag-Leffler theorem to establish the existence of some elements previously constructed by complicated calculations using infinite products. Some parts of the new proof can be found in [15] and [35]. We also determine exactly the points at which the continuum hypothesis is required.

As a sample, let me sketch the proof that *(i)* implies *(iii)* in Theorem 3.1. The result is [14, Theorem 4.2], but the present much simpler proof is [15, Theorem 5.1].

We recall the Mittag-Leffler theorem. Let (E_n) be a sequence of complete metric spaces, and let $f_n: E_{n+1} \to E_n$ be a continuous map. Suppose that $f_n(E_{n+1})$ is dense in E_n for each n. Then $\cap\{f_1 \circ \cdots \circ f_n(E_{n+1}): n \in \mathbb{N}\}$ is dense in E_1.

3.2 Theorem. *Let* R *be a commutative radical Banach algebra, and suppose that* R *contains a non-zero element* a *with* $a \in \overline{a^2R}$. *Then* R *contains a non-zero* *semi-group over* $\mathbb{Q}^+_0$.

Proof. Let $R_1 = aR$, and set $|||b||| = \inf\{||c|| : ac = b\}$ for $b \in R_1$. Then $(R_1, |||.|||)$ is a radical Banach algebra. Let R_2 be the closure of a^2R_1 in $(R_1, |||.|||)$. Then R_2 is a radical Banach algebra, and $\overline{a^2R_2} = R_2$. Thus, we can suppose that $\overline{a^2R} = R$.

$$\text{Set } E = \{x \in R : \overline{xR} = R\} = \{x \in R : a^2 \in \overline{xR}\}.$$

Since

$$E = \bigcap_p \{x \in R : \inf_{y \in R} ||a^2 - xy|| < 1/p\},$$

E is a G_δ-set in R, and so E is itself a complete metric space. We check that E is stable under products.

Let $G = \text{Inv } R^\#$, the set of invertibles of $R^\#$. Then $G = \exp R^\#$ and $\bar{G} = R^\#$ because R is radical. Since $a^{n+1} \in E$ $(n \in \mathbb{N})$, $R = \overline{a^{n+1}R} = \overline{a^{n+1}G} = \bar{E}$, and so $a^{n+1}G$ is dense in E.

Set $f_n(x) = x^{n+1}$ $(x \in E,\ n \in \mathbb{N})$. Then $f_n : E \to E$ is continuous. Take $u \in a^{n+1}G$. Then $u = a^{n+1}\exp v$ for some $v \in R^\#$, and so $u = (a \exp(v/(n+1)))^{n+1} \in f_n(E)$. Thus, f_n has dense range. By the Mittag-Leffler theorem, there exists $(x_n) \subset E$ such that $x_n = x_{n+1}{}^{n+1}$ $(n \in \mathbb{N})$. In particular, $x_p = x_q{}^{q!/p!}$ $(p,q \in \mathbb{N})$. Set $\psi(r/s) = x_s{}^{r(s-1)!}$. It can be checked that $\psi(r/s)$ is well defined for each positive rational r/s, and that $r/s \mapsto \psi(r/s)$ is the required semi-group.

4. THE ROLE OF THE CONTINUUM HYPOTHESIS

The proof of the existence of discontinuous homomorphisms required the assumption of the continuum hypothesis. It is a remarkable fact that this hypothesis cannot be dropped. (We work in ZFC.)

I believe that the history of this part of the story is as follows. A seminar on the (then open) problem of the existence of discontinuous homomorphisms from $C(X)$ was given at Caltech in 1975. An undergraduate attending the seminar, Hugh Woodin, proved that, if there is a discontinuous homomorphism, then there is a certain set-theoretic consequence. Then R. Solovay constructed a model of ZFC (in which CH is false) in which this consequence did not hold, and thus gave a model of ZFC in which all homomorphisms from each $C(X)$ are continuous. I have not seen any details of this construction, and, as far as I know, it has not been published.

Subsequently, Woodin himself gave a different approach to the construction of such a model. He uses a model in which Martin's axiom (MA) holds, modifying it to ensure that there are no discontinuous homomorphisms from $C(X)$. My source for this work is Woodin's Berkeley thesis ([34]); again, as far as I know, the results have not been published elsewhere.

It seems to me that analysts and logicians may be rather timorous about the supposed arcana of each other's craft. This mutual apprehension may be excessive, and I wish to build a bridge between the two areas, or at least an approach road to one end of the bridge.

Thus, for the remainder of this section, we work in ZFC, but we do not assume CH without specific mention.

First, we determine the extent to which the problems for different X are equivalent. We write ℓ^∞ for the algebra of all real-valued, bounded sequences, and c_0 for the real-valued sequences convergent to 0, on $\mathbb{N}$, so that ℓ^∞ is isomorphic to $C_{\mathbb{R}}(\beta\mathbb{N})$.

4.1 Theorem.

(i) If there is a discontinuous homomorphism from ℓ^∞, *then there is a discontinuous homomorphism from* $C(X)$ *for each infinite* X.

(ii) If there is a discontinuous homomorphism from any $C(X)$, *then there is a discontinuous homomorphism from* c_0.

Proof. *(i)* This is easy because each infinite topological space contains a countably infinite, discrete subset: see [10, 7.8].

(ii) ([22]) First note that there is a separable C*-subalgebra of $C(X)$ which is the domain of a discontinuous homomorphism, and so we may suppose that $C(X)$ is separable, and hence that X is metrizable, with metric d, say.

We know from § 1 that there is a finite set $F \subset X$, $p \in \beta(X\setminus F)\setminus(X\setminus F)$, and a non-zero homomorphism ν from $C_o(X\setminus F)$ with $\ker \nu \supset J_p^{\beta}$. Set $Y = X\setminus F$.

For $n \in \mathbb{N}$, set

$$G_n = \{x \in X : \frac{1}{2n+3} < d(x,F) < \frac{1}{2n}\} ,$$

and set $G_0 = \{x \in X : d(x,F) > 1/3\}$. Let $U_1 = \cup\{G_{2n} : n = 0,1,\dots\}$, $U_2 = \cup\{G_{2n+1} : n = 0,1,\dots\}$, so that U_1 and U_2 are open sets in Y with $U_1 \cup U_2 = Y$. Let $V_j = \beta Y\setminus\overline{(Y\setminus U_j)}$ for $j = 1,2$, where the bar denotes closure in βY. Then V_1 and V_2 are open sets in βY with $V_1 \cup V_2 = \beta Y$. We may suppose that $p \in V_1$. Note that, if $f \in C_o(Y)$ and $f = 0$ on U_1, then $\hat{f} = 0$ on V_1, a neighbourhood of p, and so $\nu(f) = 0$.

Take $f_n \in C(Y)$ with $f_n = 1$ on G_n, with $f_n = 0$ on G_m for $m \notin \{n-1, n, n+1\}$, and with $f_n(Y) \subset [0,1]$. For each $a = (\alpha_n) \in c_o$, set

$$\psi(a)(x) = \sum_{n=1}^{\infty} \alpha_n f_{2n}(x) \quad (x \in Y).$$

Each $x \in Y$ has a neighbourhood on which at most two of the functions f_{2n} are non-zero, and so $\psi(a) \in C(Y)$. Since $\alpha_n \to 0$, $\psi(a) \in C_o(Y)$, and clearly $\psi: a \mapsto \psi(a)$ is linear. The map ψ is not necessarily a homomorphism. However, on the set U_1, $f_{2n}{}^2 = f_{2n}$ and $f_{2n}f_{2m} = 0$ $(n \neq m)$, and so $\nu(\psi(ab) - \psi(a)\psi(b)) = 0$ for $a,b \in c_o$. Thus, $\nu \circ \psi: c_o \to R$ is a homomorphism.

Clearly, $(\nu \circ \psi)(a) = 0$ if a is eventually zero. We must show that $\nu \circ \psi \neq 0$.

Take $g \in C_o(Y)$ with $\nu(g) \neq 0$, and let $\beta_n = \sup\{|g(x)|: x \in G_n\}$. Then $(\beta_n) \in c_o$, and there exists $a = (\alpha_n)$, $(\gamma_n) \in c_o$ with $\alpha_n \neq 0$ and $\alpha_n\gamma_n = \beta_n$ $(n \in \mathbb{N})$. Let $f = g/\alpha_n$ on $\bar{G}_{2n}$, and set $f = 0$ on F. Then f is continuous on $\cup \bar{G}_{2n} \cup F$. Extend f to belong to $C_o(Y)$. Then $\psi(a)f = g$ on U_1, and so $\nu(\psi(a)f) = \nu(g)$. Thus, $(\nu \circ \psi)(a) \neq 0$, as required.

It is not known (in ZFC) whether or not the existence of a discontinuous homomorphism from c_o implies the existence of such a map from ℓ^{∞}, and this seems to be an interesting question.

Let Y be a completely regular space. Then there is a bijection between the points of βY and the z-ultrafilters on Y (see [17]). If Y is discrete, then each z-ultrafilter on Y is an ultrafilter - and in particular this is true for $Y = \mathbb{N}$. Let $p \in \beta\mathbb{N}$, and let

$$\mathcal{U}_p = \{U \cap \mathbb{N} : U \text{ is a neighbourhood of } p \text{ in } \beta\mathbb{N}\}.$$

Then the map $p \mapsto \mathcal{U}_p$ is a bijection from $\beta\mathbb{N}$ onto the set of ultrafilters on $\mathbb{N}$; the points of $\beta\mathbb{N}\setminus\mathbb{N}$ correspond to the free ultrafilters.

From the above remarks, we see that there is a discontinuous homomorphism from c_0 (respectively, ℓ^∞) if and only if there is a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and a non-zero homomorphism $\nu : c_0/\mathcal{U} \to R$ (respectively, $\ell^\infty/\mathcal{U} \to R$), for some radical Banach algebra R.

I now give the key result of Woodin. As a gesture towards logicians' mores, we write ω for the ordinal $\{0,1,2,\ldots\}$, ω^ω for the set of functions from ω to ω, and we take an ultrafilter $\mathcal{U}$ on ω, corresponding to a point, say p, of $\beta\omega\setminus\omega$.

First, we require some further notation. For $f,g \in \omega^\omega$, we define:

$f < g$ if $f(n) < g(n)$ eventually;

$f \equiv g$ if $f(n) = g(n)$ eventually;

$f <_{\mathcal{U}} g$ if $\{n \in \omega : f(n) < g(n)\}$ belongs to $\mathcal{U}$.

$f \equiv_{\mathcal{U}} g$ if $\{n \in \omega : f(n) = g(n)\}$ belongs to $\mathcal{U}$.

The order $<$ on ω^ω is the Fréchet order. We write $\mathrm{ult}_{\mathcal{U}}\omega$ for the family of cosets of ω^ω with respect to the equivalence relation $\equiv_{\mathcal{U}}$, $[f]$ for the coset of an element $f \in \omega^\omega$, and $<_{\mathcal{U}}$ for the induced order on $\mathrm{ult}_{\mathcal{U}}\omega$. For $g \in \omega^\omega$, let

$$\mathrm{ult}_{\mathcal{U}} g = \{a \in \mathrm{ult}_{\mathcal{U}}\omega : a <_{\mathcal{U}} [g]\}.$$

Let A be an algebra, and let $a,b \in A$. Then $b << a$ if $a \in bA$. A map $\pi: \mathrm{ult}_{\mathcal{U}} g \to \omega^\omega$ is isotonic if $\pi(a) < \pi(b)$ in ω^ω whenever $a <_{\mathcal{U}} b$ in $\mathrm{ult}_{\mathcal{U}} g$.

4.2 Theorem. (Woodin) *Suppose that there is a discontinuous homomorphism from some* $C(X)$. *Then there is an unbounded, monotone increasing function* g *in* ω^ω, *a free ultrafilter* $\mathcal{U}$ *on* ω, *and an isotonic injection* $\pi: \mathrm{ult}_{\mathcal{U}} g \to \omega^\omega$.

Proof. Let $\nu: c_0/\mathcal{U} \to R$ be a non-zero homomorphism, as above. We construct π as a composition:

$$\mathrm{ult}_{\mathcal{U}} g \xrightarrow{\sigma} c_0/\mathcal{U} \xrightarrow{\nu} R \xrightarrow{\tau} \omega^\omega.$$

First we define g. Choose $a = (\alpha_n) \in c_0$ such that $\nu([a]) \neq 0$: here, $[a]$ is the coset of a in $c_0/\mathcal{U}$. We can suppose that $0 < \alpha_n < 1$ $(n \in \omega)$. Then choose an unbounded, monotone $g \in \omega^\omega$ with $\lim \alpha_n^{1/g(n)^2} = 0$.

Take $[f] \in \mathrm{ult}_{\mathcal{U}} g$. We may suppose that $f(n) < g(n)$ $(n \in \omega)$. Set

$$\sigma([f]) = [(\alpha_n^{f(n)/g(n)^2})].$$

If $[f_1] \prec_{\mathcal{U}} [f_2] \prec_{\mathcal{U}} [g]$, let

$$\beta_n = \alpha_n^{(f_2(n) - f_1(n))/g(n)^2}.$$

Then $(\beta_n) \in c_0$, and $\sigma([f_1])[(\beta_n)] = \sigma([f_2])$, so that $\sigma([f_1]) \ll \sigma([f_2])$ in $c_0/\mathcal{U}$. Thus, $(\nu \circ \sigma)(a) \ll (\nu \circ \sigma)(b)$ in R whenever $a \prec_{\mathcal{U}} b$.

Secondly, we check that the range of $\nu \circ \sigma$ is contained in the non-nilpotents of R. For suppose that $[f] \prec_{\mathcal{U}} [g]$ and that $r \in \mathbb{N}$. Set

$$\beta_n = \alpha_n^{1-(rf(n)/g(n)^2)}.$$

Then eventually $1 - (rf(n)/g(n)^2) > 1 - (r/g(n)) > 1/2$, and so $\beta_n < \alpha_n^{1/2}$ eventually. Thus, $(\beta_n) \in c_0$. Also, $\beta_n\alpha_n^r = \alpha_n$, and so $(\nu \circ \sigma)([f]^r) \neq 0$, as required.

Finally, if x is a non-nilpotent element of R, set $\tau(x)(n) = [\|x^n\|^{-1}]$, where now $[\zeta]$ denotes the integral part of a real number ζ. If $x = yz$ in R, where x,y,z are non-nilpotent, then

$$[\|y^n\|^{-1}][\|z^n\|^{-1}] < [\| x^n\|^{-1}].$$

Since R is radical, $\|z^n\| \to 0$ as $n \to \infty$, and so $\tau(y) < \tau(x)$ in ω^ω.

The result follows.

Let $(S, <)$ be a partially ordered set. A chain in S is a totally ordered subset. Elements $r,s \in S$ are incompatible if there is no element $t \in S$ with $t < r$ and $t < s$. An antichain in S is a subset T of S such that, if $r,s \in T$ with $r \neq s$, then r and s are incompatible. The set S has the countable chain condition if each antichain is countable.

A subset T of S is dense if, for each $s \in S$, there exists $t \in T$ with $t \prec s$. A subset T is a filter if (i) for each $r,s \in T$, there exists $t \in T$ with $t \prec r$ and $t \prec s$, and (ii) for each $r \in T$ and each $s \in S$ with $s \prec t$, we have $s \in T$.

We now come to Martin's Axiom (MA). This is the statement : if $(S,\prec)$ is a non-empty, partially ordered set satisfying the countable chain condition, and if $\mathcal{D}$ is a collection of dense subsets of S such that $\mathcal{D}$ has cardinality at most ω_1, then there is a filter T in S such that $T \cap D \neq \emptyset$ for each $D \in \mathcal{D}$.

For an account of MA, see Kunen's book [24, II, § 2]. Some applications of MA to general topology, illuminating for the analyst, are given in [24, II, § 3].

The theorems proved by Woodin in [34] which are required for the result about $C(X)$ are the following.

4.3 Theorem. *There is a model* M *of set theory with the following properties: (i)* MA *holds; (ii)* CH *fails; (iii) the set* $(2^{\omega_1}, \prec)$ *does not embed in* $(\omega^{\omega}, \prec)$.

Here, $(2^{\omega_1},\prec)$ is the set of functions from ω_1 into the set $\{0,1\}$ with the lexicographic order; an embedding must preserve order. In the model, $2^{\aleph_0} = \aleph_2$, and it is not clear to me just what values $2^{\aleph_0}$ can have in similar models. The proof of 4.3 is a variant of the proof that there are models in which MA holds: for this, see [24, VIII, § 6].

4.4 Theorem. *In* ZFC, MA + ¬ CH *implies that, for each free ultrafilter* U *over* ω *and for each unbounded, monotone increasing function* g *in* ω^{ω}, $(2^{\omega_1}, \prec)$ *does embed in* $\mathrm{ult}_U g$.

This is [34, 7.2]. Theorems 4.2, 4.3, and 4.4 together show that, in the model M of 4.3, there are no discontinuous homomorphisms from any $C(X)$ into any Banach algebra. We hope to present further details of these results, in terms comprehensible to analysts, in [11].

To date, no one has demonstrated that there are models in which there are discontinuous homomorphisms from c_0, but in which CH fails. Thus, the present status of the statement "all homomorphisms from each $C(X)$ are continuous" is that it is independent of ZFC, and implies that $2^{\aleph_0} > \aleph_1$.

5. OPEN QUESTIONS

The results of the previous sections show that all homomorphisms from certain Banach algebras are continuous, and that there are examples of discontinuous homomorphisms from some C*-algebras. (We now assume that CH holds.) However, a number of basic points remain open.

We consider first the following well-known question.

Question 1. *Let* $\theta: A \to B$ *be a homomorphism between Banach algebras, and suppose that* $\overline{\theta(A)} = B$. *If* B *is semi-simple, is* θ *automatically continuous?*

If this were true, it would be a striking generalization of Gelfand's result, when B is commutative, and Johnson's result, when $\theta(A) = B$.

It is not difficult to see ([1, 1.1]) that the question is equivalent to the following. Let $\theta: A \to B$ be a homomorphism with separating space $S(\theta)$. Is $S(\theta) \subset Q(B)$, the set of quasi-nilpotents of B?

In general, we should like any new information about $\sigma(b)$, the spectrum of b, for $b \in S(\theta)$. It is easy to show that $\sigma(b)$ is always a connected subset of $\mathbb{C}$ containing the origin ([32, 6.16]), but this seems to be all that is known in general. Of course, by (1), $S(\theta) \subset Q(B)$ if B is commutative, and, by (2), we always have

$$S(\theta) \cap \theta(A) \subset Q(B), \tag{5}$$

but it does not seem to be easy to modify Aupetit's proof to obtain the stronger result that $S(\theta) \subset Q(B)$. Note that the spectral radius ν is not necessarily a continuous function on a Banach algebra (see [3, Chap. 1, § 5]).

Some partial results on Question 1 are given in [1, § 3].

Let us now consider Question 1 in the case that both A and B are C*-algebras. Then $\overline{I(\theta)}$ and $S(\theta)$ are closed ideals in A and B, respectively, and so both are themselves C*-algebras. If θ is not continuous, then, by Theorem 2.5, there is a non-zero homomorphism $\nu: \overline{I(\theta)} \to S(\theta)$. Clearly, $S(\nu) = S(\theta)$ and $I(\nu) = I(\theta)$. Also, $\nu(\overline{I(\nu)})$ is dense in $S(\nu)$, and so, by (5), $S(\nu)$ contains a dense subalgebra of quasi-nilpotents. Thus, we have the following result, pointed out to me by Kjeld Laursen.

5.1. Theorem. *Suppose that there is a discontinuous homomorphism* $\theta: A \to B$, *where* A *and* B *are* C*-*algebras and* $\overline{\theta(A)} = B$. *Then there is a non-zero* C*-*algebra* $\mathcal{A}$ *and a dense subalgebra* $\mathcal{B}$ *such that each element of* $\mathcal{B}$ *is a quasi-nilpotent.*

Note that $\mathcal{B}$ is not just a dense subset of $\mathcal{A}$ consisting of quasi-nilpotents, but a subalgebra. No such C*-algebra $\mathcal{A}$ is known, and I hope someone can show me that no such algebra can exist.

The second question is the following.

Question 2. *For which* C*-*algebras* A *is it true that each homomorphism from* A *into a Banach algebra is automatically continuous?*

We know that the C*-algebras B(H) and K(H) belong to the class, but that C(X) does not belong to the class for each infinite X.

Let A be a C*-algebra. A representation of dimension n of A is a $\star$-homomorphism π from A into $M_n(\mathbb{C})$, the algebra of n×n matrices. The representation is irreducible if {0} and $M_n(\mathbb{C})$ are the only linear subspaces of $M_n(\mathbb{C})$ which are invariant for $\pi(A)$. Two irreducible representations of dimension n are equivalent if they have the same kernel.

The following result is [1, 2.5].

5.2 Theorem. *Let* A *be a* C*-*algebra. Suppose that, for some* $n \in \mathbb{N}$, A *has infinitely many non-equivalent irreducible representations of dimension* n. *Then there is a discontinuous homomorphism from* A *into a Banach algebra.*

We made the guess in [1] that, if A does not satisfy the condition in Theorem 5.2, then all homomorphisms from A are continuous. Maybe this is too simple a conjecture.

The only way I know to prove that homomorphisms from a C*-algebra A are continuous is to combine the methods described in § 2, above. This is done in [1, § 4]. We consider a class of C*-algebras which we call the AW*M-algebras, and we prove that, if A is an AW*M-algebra, then there is a discontinuous homomorphism from A if and only if A has infinitely many non-equivalent irreducible representations of dimension n for some n. The description of the class of AW*M-algebras is rather technical. Let me just note the class contains:

(i) each AW*, and hence each von Neumann, algebra; (ii) each closed ideal in an AW*-algebra; (iii) each commutative C*-algebra.

However, many C*-algebras are not of this class. It seems likely that a new idea will be necessary to resolve the question for these algebras.

Here is one example of a class of C*-algebras for which the answer is not known.

Let A be a simple, infinite-dimensional C*-algebra without identity, and let $\theta: A \to B$ be a homomorphism. Then $\overline{I(\theta)} = A$, but we cannot conclude that $I(\theta)$ is closed. Let $M(A)$ be the multiplier algebra of A. Then $\overline{I_M(\theta)}$ has finite codimension in $M(A)$. However, there are examples ([30]) of simple C*-algebras A such that A has finite codimension in $M(A)$. I cannot see how to exclude the possibility that $A = \overline{I(A)} = \overline{I_M(A)}$, but $A \neq I(A)$, in this example.

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EPIMORPHISMS OF C*-ALGEBRAS

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The question of continuity of homomorphisms mapping C*-algebras onto Banach algebras is discussed. The developments leading up to Esterle's positive solution for commutative C*-algebras are described. The general question is then considered in two stages: one where restrictions are imposed on the domain algebra and one where commutativity is retained in the range. As a natural consequence of this latter case the existence question for discontinuous homomorphisms with commutative range is discussed.

PROLOGUE

This is an extended version of a lecture entitled "Commutative homomorphisms of C*-algebras" delivered at the Third Paderborn Conference on Functional Analysis.

At the conference this lecture was preceded by Garth Dales' widespanning survey of automatic continuity of homomorphisms from C*-algebras and I benefited greatly from this, being able to draw on concepts and results already mentioned by Dales.

It seems reasonable to make an analogous approach here. Therefore, I suggest that if you, dear reader, have not already done so, please read [8]. At the very least, take a look at [8] up to and including theorem 1.3.

Should this suggested study plan be met with suspicion, perhaps even a stubborn decision not to follow my advice, no harm need result: what follows will be largely selfcontained, albeit perhaps a bit laconic when it comes to motivational remarks.

I. THE $C(X)$ PROBLEM AND ITS SOLUTION.

In contrast with the river Pader which wells from numerous springs, automatic continuity is generally considered as stemming from one source, namely Kaplansky's observation [17] that any algebra epimorphism of $C(X)$ is norm-increasing (some people would take issue with the single-source claim, with reference to early work on the uniqueness of norm topology. However, the automatic continuity theory started as a study of commutative Banach algebras and the main motivation was clearly Kaplansky's question).

Kaplansky's result was that if $C(X)$ is the Banach algebra (with sup norm $|\cdot|_X$) of continuous complex valued functions on the compact Hausdorff space X and if $\theta: C(X) \to B$ is an algebra monomorphism (i.e. θ is 1-1) into the Banach algebra B then for any $f \in C(X)$:

$$\|\theta(f)\| \geq |f|_X \quad .$$

Evidently this means (by the open mapping theorem) that if θ is surjective then $\| \|$ and $|\cdot|_X$ are comparable. This shows that $C(X)$ has a unique complete norm topology. It also made Kaplansky ask the very natural question: can $C(X)$ be equipped with another (i.e. inequivalent, hence necessarily incomplete) algebra norm ?

So Kaplansky's answer was for the case of a homomorphism which is surjective and 1-1, and Kaplansky's question consisted of dropping the assumption of surjectivity. Dropping also the 1-1 assumption clearly brought up nothing new: there exists a discontinuous homomorphism on $C(X)$ if and only if there is a discontinuous monomorphism of $C(X)$; if $\theta: C(X) \to B$ is discontinuous, simply consider $\theta \oplus \text{identity}: C(X) \to B \oplus C(X)$.

The other possibility was to retain surjectivity and drop 1-1. In other words: if $\theta: C(X) \to B$ is our epimorphism is θ necessarily continuous? As it turned out, it took longer to answer this than to answer Kaplansky's question. But then how much attention does an unposed question ordinarily receive? We shall return to the surjective case in section III but first develop some of the general tools that allowed the important reductions which eventually permitted the Dales-Esterle solutions of Kaplansky's question: the reduction to the radical range and the prime kernel case.

Both Dales [7] and Esterle [9] answered Kaplansky's question by showing that if J is a prime ideal with closure M in $C(X)$ then there is a monomorphism of M/J into some radical commutative Banach algebra R. Their answers require the assumption of the continuum hypothesis (cf. [8, section 4]).

At the time (around 1974) when Dales and Esterle started looking for a discontinuous homomorphism from $C(X)$ it was known that the above "M/J-into-R" setting was exactly the place where discontinuous homomorphisms would have to be found, if indeed they existed.

This significant reduction of the scope of the search was brought about largely by Bade and Curtis [3] (cf. [8, theorem 1.3]) and Sinclair [19]. Almost everything in this reduction can be derived from one single principle, a stability result about separating spaces. And since this also leads naturally to Esterle's solution [11] of the surjectivity problem, I should like to describe this in some detail.

II. STABILITY OF SEPARATING SPACES.

Recall that if $S: X \to Y$ is a linear map from the Banach space X to the Banach space Y then the *separating space* of S is

$$\zeta(S) = \{y \in Y \mid \text{for some sequence } x_n \to 0 \text{ in } X, \; Sx_n \to y\} .$$

The closed graph theorem tells us that S is continuous exactly when $\zeta(S) = \{0\}$.

If $X = X_0$, if $\{X_n\}_{n=1}^{\infty}$ is a sequence of Banach spaces and if

$$T_n : X_n \to X_{n-1}$$

is a sequence of continuous linear maps then

$$ST_1 \dots T_{n-1}T_n : X_n \to Y$$

has separating spaces which we may denote

$$\zeta_n := \zeta(ST_1 \dots T_n) \qquad n = 1,2,\dots \; .$$

Clearly $\zeta \supseteq \zeta_1 \supseteq \dots \supseteq \zeta_n \supseteq \dots$. The surprising thing is that such a sequence always satisfies a descending chain condition:

The stability lemma. With the above assumptions there is an integer N such that

$$\zeta_N = \zeta_n \quad \text{for all } n \geq N .$$

This principle was brought into automatic continuity by Johnson and Sinclair [15] but the idea of the proof is much older [16]. This version is from [18]. The proof is short and simple enough to include here.

Proof: We need first the general observation that if $R: Y \to Z$ is a continuous linear map to the Banach space Z then $\zeta(RS) = R\zeta(S)^-$. The proof of this is not hard; it may be found in [21, lemma 1.3].

If no N can be found for which $\zeta_N = \zeta_n$ for all $n \geq N$ then, by grouping together some of the T_n's, if necessary, we may suppose that the sequence of separating spaces is strictly decreasing:

$$\zeta \neq \zeta_1 \neq \zeta_2 \neq \dots \ .$$

Let $Q_n : Y \to Y/\zeta_n$ be the natural quotient maps (each ζ_n is a closed linear subspace of Y). By the first paragraph of this proof and the assumption of strict decrease we see that $Q_n S T_1 \dots T_m$ is discontinuous for $m < n$ and continuous for $m \geq n$.

We may as well suppose that $\|T_m\| = 1$ for each m; we can then select inductively (y_n) (for $n \geq 2$) such that

$$\|y_n\| < 2^{-n} \quad \text{and}$$

$$\|Q_n S T_1 \dots T_{n-1} y_n\| \geq n + \|Q_n S T_1 \dots T_n\| + \sum_{p=2}^{n-1} \|Q_n S T_1 \dots T_{p-1} y_p\| \ .$$

For the element $x_0 = \sum_{n=2}^{\infty} T_1 \dots T_{n-1} y_n$ we then have, for each $n \geq 2$,

$$\|Sx_0\| \geq \|Q_n S x_0\| = \|Q_n S \sum_{m=2}^{n-1} T_1 \dots T_{m-1} y_m + Q_n S T_1 \dots T_{n-1} y_n + Q_n S \sum_{m>n} T_1 \dots T_{m-1} y_m\|$$

$$\geq n + \|Q_n S T_1 \dots T_n\| + \sum_{p=2}^{n-1} \|Q_n S T_1 \dots T_{p-1} y_p\|$$

$$- \sum_{m=2}^{n-1} \|Q_n S T_1 \dots T_{m-1} y_m\| - \|Q_n S T_1 \dots T_n\| \, \|\sum_{m>n} T_{n+1} \dots T_{m-1} y_m\| \geq n \ .$$

Corollary. (Semigroup stability), [21]. Suppose $\{T^\alpha\}_{\alpha \in \mathbb{R}_+}$, $\{R^\alpha\}_{\alpha \in \mathbb{R}_+}$ are semigroups of continuous linear operators on the Banach spaces X and Y, respectively, which intertwine with the linear map $S: X \to Y$, (i.e. $R^\alpha S = S T^\alpha$, $\alpha \in \mathbb{R}_+$). With

$$\zeta_\alpha := \zeta(S T^\alpha) \qquad (\alpha \in \mathbb{R}_+)$$

we have

$$\zeta_\alpha = \zeta_\beta \qquad (\alpha, \beta \in \mathbb{R}_+) \ .$$

Proof: It is not difficult to see (cf. the beginning of the proof of the stability lemma) that $\zeta_\alpha = [R^\alpha \zeta(S)]^-$ $(\alpha \in \mathbb{R}_+)$.

Suppose first that for certain $\alpha < \beta$ we have $\zeta_\alpha = \zeta_\beta$. Then $\zeta_{\alpha+n(\beta-\alpha)} = \zeta_\alpha$ for $n = 1,2,\dots$ follows by induction: The case $n = 1$ is assumed. Since

$$\zeta_{\alpha+(n+1)(\beta-\alpha)} = [R^{\alpha+(n+1)(\beta-\alpha)} \zeta(S)]^-$$

$$= [R^{\beta-\alpha} R^{\alpha+n(\beta-\alpha)} \zeta(S)]^-$$

$$= [R^{\beta-\alpha} \zeta_\alpha]^- = [R^\beta \zeta(S)]^- = \zeta_\alpha$$

the inductive claim follows.

Consequently, if $\gamma > \alpha$, then there is an integer n so that $\gamma \in [\alpha+n(\beta-\alpha),\ \alpha+(n+1)(\beta-\alpha)[$, and hence $\zeta_\gamma \supseteq \zeta_{\alpha+(n+1)(\beta-\alpha)} = \zeta_\alpha \supseteq \zeta_\gamma$.

On the other hand suppose $\exists\alpha < \beta$ with $\zeta_\alpha \neq \zeta_\beta$. Choose an infinite series $\Sigma\alpha_j$ with positive terms and sum $\leq \alpha$. If $s_n = \sum_{j=1}^{n} \alpha_j$ then $\zeta_{s_n} \neq \zeta_{s_{n+1}}$ for each n (because otherwise $\zeta_\alpha = \zeta_\beta$, by what we have already shown). This infinite descent of a sequence of separating spaces contradicts the stability lemma. The corollary follows.

Corollary. (prime ideal kernel). If $\theta\colon A \to B$ is an algebra homomorphism from the commutative unital Banach algebra A with dense range in the Banach algebra B then there is a closed ideal $K \subseteq B$ so that the homomorphism $\phi\colon A \to B \to B/K$ has prime kernel.

Proof: Let ζ be the separating space of θ and let

$$I = \{a \in A \mid \theta(a)\zeta = \{0\}\}.$$

Suppose the following situation occurs: for every $a \in A$ either $\theta(a)\zeta = \{0\}$ or $(\theta(a)\zeta)^- = \zeta$. We then let a_0 be the unit of A and note that the ideal I is prime: with $\theta(ab)\zeta = \{0\}$ and $\theta(a)\zeta \neq \{0\}$ we have $\{0\} = [\theta(ab)\zeta]^- = [\theta(b)\theta(a)\zeta]^- = (\theta(b)\zeta)^-$ (since $(\theta(a)\zeta)^- = \zeta$). Thus $b \in I$, if $ab \in I$ and $a \notin I$.

If this situation does not occur there must be an element $a_1 \in A$ for which

$$\{0\} \neq (\theta(a_1)\zeta)^- \neq \zeta.$$

Now let

$$I_1 = \{a \in A \mid \theta(aa_1)\zeta = \{0\}\}.$$

If $\theta(aa_1)\zeta = \{0\}$ or $= (\theta(a_1)\zeta)^-$ then I_1 is prime. This is shown as we just did it. If this dichotomy does not arise there must be $a_2 \in A$ for which

$$\{0\} \neq [\theta(a_2a_1)\zeta]^- \neq [\theta(a_1)\zeta]^-.$$

We may repeat this argument, but not forever: if we could, we would produce a sequence $a_1, a_2, \ldots \in A$ for which

$$\{0\} \neq (\theta(a_n\ldots a_1)\zeta)^- \neq (\theta(a_{n-1}\ldots a_1)\zeta)^-.$$

This would contradict the stability lemma.

Thus we must get an element, call it $a_0 \in A$ (a_0 is the product of the elements arising from the repetition of the above argument), for which the set

$$J = \{a \in A \mid \theta(aa_0)\zeta = \{0\}\}$$

is prime.

If we let $K = \{b \in B \mid b\theta(a_0)\zeta = \{0\}\}$ then obviously K is a closed ideal and if $\phi: A \to B \to B/K$ then $\ker\phi = \{a \in A \mid \theta(aa_0)\zeta = \{0\}\} = J$ is prime.

Note that phrased in this generality the above corollary contains no continuity claims. In the $C(X)$ case that we are primarily interested in, a few additional observations will do.

The next result compresses the work of Bade and Curtis [3] and of Sinclair [19] into one set of statements. Here X is locally compact Hausdorff.

Theorem. Let $\theta: C_0(X) \to B$ be a discontinuous homomorphism with dense range in the Banach algebra B. Then we may assume without loss of generality that B is a radical Banach algebra with an adjoined unit and that θ has a prime kernel.

Moreover, with these reductions

$$[\theta(f)B]^- = [\theta(f^n)B]^- , \quad n = 1,2,\ldots$$

for every $f \in C_0(X)$.

Proof: With the notation of the previous proof it is clear that if J is not closed then ϕ is not continuous.

If we suppose that J is closed then (since $\{0\}$ is not a prime ideal) we may suppose that J is all of $C_0(X)$ or else that J is a maximal modular ideal.

In either case, if $\theta(J)\zeta$ is dense in ζ then $\theta(J)\theta(a_0)\zeta = \{0\}$ (this is the definition of J) implies that $\theta(a_0)\zeta = \{0\}$, contrary to the definition of a_0. So $\{\theta(J)\zeta\}^-$ is a closed ideal, properly contained in ζ. Thus if

$$Q : B \to B/(\theta(J)\zeta)^-$$

is the quotient map then $Q\circ\theta : C_0(X) \to B/(\theta(J)\zeta)^-$ is a homomorphism with $\zeta(Q\circ\theta) = Q\zeta \neq \{0\}$, i.e. this map is discontinuous. But also

$$(Q\circ\theta)(J)\zeta(Q\circ\theta) = \{0\} ,$$

so $Q\circ\theta$ is continuous on J, hence continuous.

This contradiction shows that J cannot be closed.

We have obtained a discontinuous homomorphism of $C_0(X)$ with a prime kernel.

Obviously, θ must be discontinuous on J^- which is a maximal modular, hence cofinite ideal. But then it follows that

$$\zeta \subseteq \theta(J^-)^- .$$

Moreover, if $a \in J^-$ and $\{a_n\} \subset J$ with $a_n \to a$ then $\theta(a-a_n) \to \theta(a)$ and $a - a_n \to 0$, i.e. $\theta(a) \in \zeta$. It follows that

$$\zeta = \theta(J^-)^- .$$

However, it is easy to see that ζ is a radical algebra: every element must have zero spectral radius.

The last claim of this theorem comes from the semigroup stability (since we may take $B = \zeta$), once we have seen that for every $f \in C_0(X)$ $[\theta(f)\zeta]^- = [\theta(|f|^2)\zeta]^-$: since $f = a|f|^{\frac{1}{2}}$ for suitable $a \in C_0(X)$, we have that

$$[\theta(f)\zeta]^- \supseteq [\theta(f\bar{f})\zeta]^- = [\theta(|f|^2)\zeta]^- = [\theta(|f|^{\frac{1}{2}})\zeta]^- \supseteq [\theta(|f|^{\frac{1}{2}}a)\zeta]^- = [\theta(f)\zeta]^- .$$

This proves the Bade-Curtis-Sinclair result.

Algebras of the form J^-/J, J a non-closed prime of $C(X)$, are particularly rich in known structure (cf. section 3 of [8]). As I mentioned, this was the setting in which the first examples of discontinuous homomorphisms were found.

What's more, for the question of continuity of epimorphisms of $C_0(X)$ - where the answer turned out to be that they are all continuous [11] - this reduction is perhaps even more significant. To understand how Esterle's solution works we need one additional observation about the primes of the quotient J^-/J: in this ring the primes are totally ordered by the usual inclusion order. The real case is classical [14] and the extension to the complex case is routine.

III. EPIMORPHISMS

In [11] was proved this result. The proof was cumbersome, but later Esterle [12] has given a much shorter soft one. This we can give here.

Theorem. If B is a radical commutative Banach algebra in which there is a non-nilpotent element $b \in (bB)^-$, then the primes of B are not linearly ordered (by inclusion).

Proof: Let $I = (bB)^-$. A lengthy, and for our purposes not particularly enlightening inductive argument from [11, lemma 4.4] will produce a sequence of invertible elements (s_n) from $B^{\#}$ (B with an adjoined unit) for which

$s_n b \to b$ and $\|s_n^{-1} b^p\| \to \infty$ as $n \to \infty$, for each $p = 1,2,\ldots$.

If we then define, for fixed integers n,p:

$$\Omega_{n,p} = \{(c,d) \in I\times I \mid \exists x \in B^{\#} : \|cx-x\| + 1/\|d^p x\| < \tfrac{1}{n}\}$$

then $\Omega_{n,p}$ is obviously open. It turns out to be also dense in $I\times I$:

Let $\varepsilon > 0$ be given and let $bx, by \in I$.

There is no harm in replacing y by $y+\varepsilon$, and hence we may suppose y invertible in $B^{\#}$. Now choose n so that $1/2n < \varepsilon$ and then s_m so that

$$\|bxs_m - bxs_m s_m^{-1}\| = \|bxs_m - bx\| < \frac{1}{2n} < \varepsilon$$

and so that

$$\|(by)^p s_m^{-1}\| > 2n .$$

The first line is easily obtained. The second one follows from

$$\|b^p s_m^{-1}\| \le \|y^{-p}\| \, \|(by)^p s_m^{-1}\| .$$

This shows that $(bxs_m, by) \in \Omega_{n,p}$ and since $\|bxs_m - bx\| < \varepsilon$, while $\|b(y+\varepsilon) - by\| = \varepsilon\|b\|$ we have shown that $\Omega_{n,p}$ is dense in $I\times I$.

The Baire category theorem then shows that $\Omega = \bigcap_{n,p} \Omega_{n,p}$ is a dense G_δ and if $\Omega' = \{(d,c) \mid (c,d) \in \Omega\}$ then Ω' (being a homeomorphic image of Ω) must also be a dense G_δ. All in all, again by Baire, $\Omega \cap \Omega'$ is a dense G_δ set in $I\times I$. All we need to know is that $\Omega \cap \Omega'$ is non-empty. Thus if $(c,d) \in \Omega \cap \Omega'$ and if $p \in \mathbb{N}$ then we can find sequences (a_n), (b_n) in $B^{\#}$ for which $\|ca_n - c\| + \|db_n - d\| + 1/\|d^p a_n\| + 1/\|c^p b_n\| \to 0$. This means that $d^p \notin cB^{\#}$ and $c^p \notin dB^{\#}$ (easily checked). Thus there are primes P_c, P_d of $B^{\#}$ such that $c \in P_c \setminus P_d$ and $d \in P_d \setminus P_c$. The algebra B is radical so P_c, P_d are ideals in B. This proves the theorem.

This is all we need for Esterle's theorem on the automatic continuity of epimorphisms from $C_0(X)$.

Theorem. [11, theorem 5.3]. Any epimorphism of $C_0(X)$ onto a Banach algebra is necessarily continuous.

Proof: If $\theta: C_0(X) \to B$ is a discontinuous epimorphism then we may assume that B is radical, that $\ker\theta$ is a prime and that $[\theta(f)B]^- = [\theta(f)^n B]^-$ for each $f \in C_0(X)$ and $n = 1,2,\ldots$. If $f \notin \ker\theta$ then this means that $\theta(f)$ is non-nilpotent. Also $\theta(f)^2 \in (\theta(f)^2B)^-$ and thus by Esterle's theorem the primes of B are not totally ordered. But in $C_0(X)/\ker\theta$ the primes are totally ordered. This contradiction shows that θ cannot be discontinuous.

We all know that $C_0(X)$ can be thought of as a commutative C*-algebra and thus Esterle's beautiful solution of the epimorphism problem naturally brings up this question: what can be said about epimorphisms from general C*-algebras? First of all, Kaplansky's original result, that the sup-norm on $C_0(X)$ is minimal among all algebra norms on $C_0(X)$, turns out to have an analogue in the general, non-commutative case. This was discovered by Cleveland [5] who showed that if θ is an isomorphism of a C*-algebra A then there is a constant M such that $\|x\| \leq M\|\theta(x)\|$ for all $x \in A$. Consequently, once again, if θ is also an epimorphism (i.e. onto a Banach algebra) then θ is necessarily continuous.

If we drop the 1-1 requirement we are left with an open problem. However, quite a few special cases have been settled. They fall naturally into two batches: those where additional assumptions are imposed upon the domain C*-algebra and those where restrictions apply to the range.

The best, known results for domains from subclasses of C*-algebras have been proved recently by Albrecht and Dales [1]. They study a class of C*-algebras, whose defining properties are too technical to warrant mention here (suffice it to say that this class contains all AW*-algebras and hence all von Neumann algebras; it also contains all commutative C*-algebras), but which have a sufficiently rich structure so that a reduction of the entire continuity question to the commutative case becomes possible. Thus, ultimately this approach relies on Esterle's result. At any rate, the conclusion is the same: any epimorphism from such a C*-algebra (an AW*M-algebra) onto a Banach algebra is automatically continuous [1, theorem 4.12].

If we drop all additional assumptions on the domain C*-algebra the simplest case to deal with presumably is that of a commutative range. And here it turns out that Esterle's original line of attack is still successful: An epimorphism of a C*-algebra onto a commutative Banach algebra is necessarily continuous.

In outline, the steps in the proof are the following: Cusack [6] shows that if a C*-algebra A is the domain of a discontinuous homomorphism then this homomorphism may be assumed to have a prime kernel (in a non-commutative algebra A a two-sided ideal I is prime if $aAb \subseteq I$ implies that $a \in I$ or $b \in I$). Moreover, surjectivity is not affected by this additional assumption.

If the closure B of the range of the discontinuous homomorphism is commutative then by essentially the same method of proof a result like the Bade-Curtis-Sinclair theorem may be proved: B may be taken to be radical and there is an non-nilpotent element b for which $b \in (bB)^-$. Consequently, by Esterle, the primes of B are not totally ordered by inclusion.

But if the prime kernel of the discontinuous homomorphism is called P and if the range B is commutative then A/P is a commutative algebra in which the primes *are* ordered (the situation in A/P is sufficiently similar to the C(X) case to permit analogous reasoning). Thus, if B and A/P are algebraically isomorphic we have once again reached the desired contradiction; the considered epimorphism must be continuous.

IV. HOMOMORPHISMS WITH COMMUTATIVE RANGE.

These last observations immediately bring a question to mind: if A is a non-commutative C*-algebra then how easy are homomorphisms with commutative range to come by - whether or not they are epimorphisms - - and whether or not continuous?

If $a,b \in A$ then $[a,b] = ab-ba$ is the commutator of a and b. If $\theta: A \to B$ is a homomorphism into the commutative Banach algebra B then clearly $\theta([a,b])=0$, whatever a and b. Thus, if we agree that I(A) is supposed to denote the *commutator ideal* of A, i.e. the smallest two-sided ideal containing all commutators, then $I(A) \subseteq \ker\theta$. And conversely, if $\theta: A \to B$ is a homomorphism with $I(A) \subseteq \ker\theta$ then the range of θ is commutative.

Consequently, to decide about the existence of (discontinuous) homomorphisms with commutative range we must know about the size of I(A) in A.

Here is an answer to this existence question. The result is new, so we include a proof.

Theorem. Let A be a separable C*-algebra with commutator ideal I(A). Then there exists a discontinuous homomorphism of A into a commutative Banach algebra if and only if I(A) is of infinite codimension in A. This result assumes the continuum hypothesis.

Proof. One direction is easy and requires no separability and no assumption of the continuum hypothesis: if $\theta: A \to B$ is discontinuous then B must contain quasi-nilpotent, non-nilpotent elements (by Sinclair's work described earlier), hence the range of θ must be infinite dimensional. But then $\ker\theta$ is of infinite codimension. Since $I(A) \subseteq \ker\theta$, the same claim holds for I(A).

For the converse, consider first the possibility that I(A) be closed. Then

$A/I(A)$ is an infinite dimensional commutative C*-algebra so by Dales' and Esterle's work [7,9] there is a commutative Banach algebra and a discontinuous homomorphism $\theta: A/J \to B$; $\ker\theta$ is non-closed. The composite map of A into B must be discontinuous, because the quotient map is open (if the composite is continuous it has a closed kernel; hence $\ker\theta$ must be closed).

We are left with the case of a non-closed commutator ideal $I(A)$.

In the next few lemmas let $\tilde{A}$ denote the smallest unital C*-algebra containing A. Note that $I(A) = I(\tilde{A})$. We begin with

Lemma 1. If $I(A)$ is not closed then there is a non-closed prime $P \subset \tilde{A}$ containing $I(A)$.

Proof: By definition the prime radical $PR(I(A))$ of $I(A)$ is the intersection of all primes containing $I(A)$. Since $\tilde{A}/I(A)$ is commutative it is not difficult to see that

$$PR(I(A)) = \{x \in \tilde{A} \mid \exists n \in \mathbb{N}: x^n \in I(A)\} .$$

If every prime containing $I(A)$ is closed then $PR(I(A))$ is closed and so contains the norm closure $I(A)^-$ of $I(A)$. This means that for every $x \in I(A)^-$ there is an integer n so that $x^n \in I(A)$. Now, by the Allan-Sinclair strengthening of Cohen factorization [2], if $a \in I(A)^-$ then there are elements $x, y_n \in I(A)^-$, such that $a = x^n y_n$, $n = 1,2,\ldots$. Hence $a \in I(A)$ so that $I(A) = I(A)^-$. This contradicts the assumption that $I(A)$ be not closed.

Corollary. If $I(A)$ is not closed then there is a prime of $\tilde{A}$ containing $I(A)$ which is not dense in $\tilde{A}$.

Proof: Let P be a non-closed prime containing $I(A)$. If P were dense P would contain invertible elements, hence equal A.

Lemma 2. If P is a non-closed prime of $\tilde{A}$ containing $I(A)$ then P is of infinite codimension.

Proof: If codim $P < \infty$ then there is an integer n so that if $x \in \tilde{A}$ we can find scalars $\alpha_1,\ldots,\alpha_n$ for which $(x-\alpha_1)\ldots(x-\alpha_n) \in P$. Since $\tilde{A}/P$ is commutative one of the factors is in P (remember: P is prime). Thus for every $x \in \tilde{A}$ there is $\alpha_x \in \mathbb{C}$ so that $x-\alpha_x \in P$. The map $x \to \alpha_x$ is linear and consequently P is of codimension ≤ 1. This means that P is closed.

Lemma 3. With P as above, the algebra $\tilde{A}/P$ contains an ideal of codimension 1.

Proof: We have seen that the norm closure P^- is a proper ideal of $\tilde{A}$; hence there is a primitive ideal M containing P^-. The ideal M/P^- is primitive in $\tilde{A}/P^-$ [22, 4.1.11 (ii)], hence of codimension 1 (because $\tilde{A}/P^-$ is commutative). But then M/P is of codimension 1 in $\tilde{A}/P$.

We now have everything we need in order to show that if A is separable and $I(A)$ is non-closed and if we assume the continuum hypothesis, then there is a discontinuous homomorphism of A into a commutative Banach algebra.

We may assume that A has a unit, i.e. may consider $\tilde{A}$ (if $\theta: \tilde{A} \to B$ is discontinuous, then θ is also discontinuous on A). Let P be a non-closed, hence infinite codimensional prime of $\tilde{A}$ containing $I(A)$. The algebra $\tilde{A}/P$ is unital and commutative and it is an integral domain. Since A is separable and since we assume the continuum hypothesis the cardinality of $\tilde{A}/P$ is $\aleph_1$. Esterle [10] has shown that under these circumstances $\tilde{A}/P$ has an algebra norm if and only if $\tilde{A}/P$ has a non-trivial character. Thus, by lemma 3, $\tilde{A}/P$ has an algebra norm. The canonical map $\tilde{A} \to \tilde{A}/P$ cannot be continuous, since P is not closed.
This completes the proof.

Remark. It follows from the above discussion that if the commutator ideal $I(A)$ is not closed then it is necessarily of infinite codimension in its closure. Incidentally, no examples of non-closed commutator ideals have been found yet.

Example. Here is a C*-algebra A with a closed commutator ideal of infinite codimension. Thus A is the domain of discontinuous homomorphisms with commutative (normed) range.

On the separable infinite dimensional Hilbert space l^2 let S be the shift

$$S: (x_1, x_2, \ldots) \to (0, x_1, x_2, \ldots)$$

and let $C^*(S)$ be the C*-algebra generated by S. It is well known that $S^*S - SS^*$ is the projection on the first coordinate and from this it follows that the compact operators $C(l^2)$ are contained in $C^*(S)$. Since $S^*S = SS^*$ modulo $C(l^2)$, the element $S + C(l^2)$ is normal in $C^*(S)/C(l^2)$ and hence $C^*(S)/C(l^2)$ is singly generated and, in fact, may be identified with the space of continuous functions on the essential spectrum of S, which is the circle group T. Since $C^*(S)/C(l^2)$ is commutative we conclude that $I(C^*(S)) \subseteq C(l^2)$ so that $I(C^*(S))$ is indeed of infinite codimension in $C^*(S)$. It may be shown, actually, (cf. e.g. [13]) that $I(C^*(S)) = C(l^2)$.

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NUCLEARITY AND FUNCTION ALGEBRAS
- A SURVEY -

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§ 1) Basic notions

(1.1) Definition: A pair (A,X) is called a function algebra if

a) X is a hemi-compact topological space $\neq \emptyset$;

b) A is a closed subalgebra of $\mathcal{C}(kX)$;

c) A contains the constants $\mathbb{C}$ and separates the points of X.

(A,X) is called a nuclear (resp. Schwartz, Montel, reflexive) function algebra, if it is a function algebra and if A as a l.c.s. is nuclear (resp. Schwartz, Montel, reflexive).

(1.2) Remarks:

"X hemi-compact" means that there exists an exhaustion of X by compact subsets $\ldots \subset K_n \subset K_{n+1} \subset \ldots$, $n \in \mathbb{N}$, s.t. for any compact $K \subset X$ there is $n_0 \in \mathbb{N}$ with $K \subset K_{n_0}$.

kX denotes the k-space associated to X, that is, a subset $U \subset X$ is open w.r. to kX iff $U \cap K$ is open in K, for all compact $K \subset X$. Hence $\mathcal{C}(kX)$(:= $\mathbb{C}$-valued continuous functions on kX) and A become Fréchet algebras with respect to the topology of compact convergence on X.

This topology is given by the sequence of seminorms

$$\|f\|_{K_n} := \sup_{x \in K_n} |f(x)|, \quad f \in \mathcal{C}(kX),$$

or in other words,

$$A = \varprojlim A_{K_n} \text{ and, in particular,}$$

$$\mathcal{C}(kX) = \varprojlim \mathcal{C}(kX)_{K_n} = \varprojlim \mathcal{C}(K_n)$$

where A_{K_n} stands for the separated comoletion of A with respect to $\|\cdot\|_{K_n}$.

The well known Gelfand theory for uniform Banach algebras generalizes to the above introduced function algebras (see [15],[19]). In particular, via the Gelfand representation

$$\Gamma : A \longrightarrow \mathcal{C}(k\sigma A),\ f \longrightarrow \hat{f},\ \hat{f}(\varphi) := \varphi(f),$$

we may identify A and the closed subalgebra $\Gamma(A) \subset \mathcal{C}(k\sigma A)$; note that the topologies of compact convergence on X resp. σA induce the homeomorphism $A \longrightarrow \Gamma(A)$. ($\sigma A$:= continuous spectrum of A endowed with the Gelfand topology). By this identification we see that (A,X) may be interpreted as a function algebra $(\Gamma(A), \sigma A)$ on its "natural" carrier space σA; both spaces are linked by the continuous evaluation map

$$j : X \longrightarrow \sigma A,\ x \longrightarrow \varphi_x ,$$

with $\varphi_x(f) := f(x),\ \forall f \in A$.

We shall often suppress X and rather conceive the "function algebra A" as the pair $(\Gamma(A),\sigma A)$. Note that $(\Gamma(A),\sigma A)$ is a natural system in the sense of Rickart ([23]).

Note that a nuclear function algebra (even a reflexive function algebra) on a non-finite space X never happens to be a uniform Banach algebra (although it is the projective limit of uniform Banach algebras). See [19],[21].

For a nuclear (resp. Schwartz) function algebra the above projective limit representation can be chosen in such a way that the canonical restriction operators

$$A_{K_{n+1}} \longrightarrow A_{K_n} ,\quad n \in \mathbb{N} ,$$

are nuclear (resp. compact) operators (simply by "thinning out" a given exhaustion $\ldots \subset K_n \subset K_{n+1} \subset \ldots$ in an appropriate way). Note that for function algebras we have:

nuclear ⇒ Schwartz ⇒ Montel ⇒ reflexive ⇒ Max Mod algebra (4.2) (Recall that a function algebra (A,X) is Montel, if A enjoys Montel's theorem on X.)

§ 2) Examples of nuclear function algebras

(2.1) Let $(X,\mathcal{O})$ be a (reduced) holomorphically separable complex analytic space (see [11]). Then the pair $(\mathcal{O}(X),X)$ is a nuclear function algebra, with $\mathcal{O}(X)$ the algebra of global holomorphic functions on $(X,\mathcal{O})$, i.e. $\mathcal{O}(X) = H^0(X,\mathcal{O})$.

Axiom (b) follows from the Grauert-Remmert theorem (see [4], p. 88). Nuclearity for $\mathcal{O}(X)$ is proved by the permanence properties of nuclear l.c.s. ([21]) and using the fact that analytic spaces can locally be modelled as analytic sets in polydisks. One may drop the holomorphical separability condition. Still $(\mathcal{O}(X), X/_{\mathcal{O}(X)})$ remains a nuclear function algebra. Recall the equivalence relation

$x \underset{\mathcal{O}(X)}{\sim} y$ iff $f(x) = f(y)\ \forall f \in \mathcal{O}(X)$.

The carrier space $X/_{\mathcal{O}(X)}$ can have rather peculiar properties. Function algebras of the above type are called holomorphic algebras. A strictly larger class of examples is obtained by taking closed subalgebras $A \subset \mathcal{O}(X)$. In [19] we construct such an algebra A which is not a holomorphic algebra (and which enjoys a lot of further curious properties). This leads to

Problem I: Let be given a holomorphic algebra $\mathcal{O}(X)$ and a closed subalgebra $A \subset \mathcal{O}(X)$. When is A a holomorphic algebra?

An outstanding subclass of holomorphic algebras is the class of Stein algebras.

A function algebra A is called a Stein algebra if there exists a Stein analytic space $(X,\mathcal{O})$ s.t. A is topologically isomorphic to $\mathcal{O}(X)$. For example, all algebras $\mathcal{O}(G)$, G a domain in $\mathbb{C}^n$, are Stein algebras, even if G is not a Stein domain (for, G possesses a Stein envelope of holomorphy $\tilde{G}$ s.t. $\mathcal{O}(G) = \mathcal{O}(\tilde{G})$).

Recall the Igusa-Remmert-Forster theorem: $(X,\mathcal{O})$ is a Stein space iff

$$j : X \longrightarrow \sigma\mathcal{O}(X)$$

is a homeomorphism. Hence a Stein algebra reconstructs its underlying Stein space! In fact, the categories of Stein algebras and Stein analytic spaces are antiequivalent (Forster's theorem). For the theory of Stein algebras cf. [6],[15],[19].

We mention a description of Stein algebras which is particularly useful for function algebraic reasonings: A function algebra is Stein if and only if it has holomorphic structure in all points of its spectrum [15],[16].

There is a more general fact with a similar proof. Let A be a function algebra which has holomorphic structure around some $\varphi \in \sigma A$. Then there is an A-convex neighborhood $U \subset \sigma A$ of φ s.t. A_U is a Stein algebra; in particular (A_U,U) is a nuclear function algebra. A more detailed discussion of this context is given in [16].

(2.2) Let E be a (DFN)-vector space and $U \subset E$ be an open subset. Denote by $\mathcal{O}(U)$ the algebra of all holomorphic functions on U, i.e. the class of all continuous functions on U which are holomorphic when restricted to $U \cap F$, for all affine complex lines $F \subset E$.

Then $\mathcal{O}(U)$ is a nuclear (F)-space with respect to the topology of compact convergence on U, by the well-known Boland-Waelbroeck theorem [3],[26]. Hence $(\mathcal{O}(U),U)$ is a nuclear function algebra. Note that the classes (2.1) and (2.2) are disjoint whenever dim $E = \infty$.

(2.3) The following class stems from a very different area. Let M be a hemi-compact $\mathcal{C}^\infty$-manifold and P an (hypo-)elliptic system of first order PDEs on M having complex $\mathcal{C}^\infty$-coefficients.

That is, a system $P_1,\ldots,P_m$ of first order partial differential operators on M s.t. for any coordinate patch $x_1,\ldots,x_n$ on M we have:

$$P_j = \sum_{k=1}^{n} a_j^k(x) \frac{\partial}{\partial x_k}, \quad 1 \le j \le m, \quad a_j^k \in \mathcal{C}^\infty(M) ;$$

it is called elliptic in x if the equalities

$$\sum a_j^k(x)\, \xi_k = 0, \quad \xi \in \mathbb{R}^n, \quad 1 \le j \le m,$$

imply $\xi=0$. It is called elliptic, if it is elliptic in all $x \in M$.
The well known regularity theorem (see [1]) states that any weak solution $u \in \mathcal{D}'(M)$ of an elliptic system P,

$$\text{i.e.,} \quad P_j(u) = 0, \quad 1 \le j \le m ,$$

is automatically in $\mathcal{C}^\infty(M)$. (More generally the regularity theorem is used to characterize hypoelliptic systems).
For (hypo-)elliptic systems we denote by $\mathcal{L}(P;M)$ the space of all solutions of P:

$$\begin{aligned}\mathcal{L}(P;M) &= \{u \in \mathcal{D}'(M) : P_j(u) = 0, \quad 1 \le j \le m\} = \\ &= \{u \in \mathcal{C}^\infty(M) : P_j(u) = 0, \quad 1 \le j \le m\} .\end{aligned}$$

(2.3.1) Proposition: Let $P = (P_1,\ldots,P_m)$ be a hypoelliptic system on M having $\mathcal{C}^\infty$-coefficients. Then $(\mathcal{L}(P,M),M)$ is a nuclear function algebra. (Instead of $\mathcal{L}(P,M)$ we should write more precisely: $\mathcal{L}(P;M_{/\mathcal{L}(P,M)})$).

Proof: By first order of the P_j the solution space is an algebra of functions on M.
Now consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(M) & \xrightarrow{P} & (\mathcal{D}'(M))^m \\ \uparrow & & \uparrow \\ \mathcal{C}^\infty(M) & \xrightarrow{P} & (\mathcal{C}^\infty(M))^m \end{array}$$

From the first line and the continuity of P we conclude that $\mathcal{L}(P,M)$ is closed with respect to the topology of compact convergence on M, from the second line and the regularity theorem we conclude that $\mathcal{L}(P,M)$, at the same time, is closed with respect to the $\mathcal{C}^\infty$-topology on M.
Hence, by the open mapping theorem for F-spaces, both topologies coincide on $\mathcal{L}(P,M)$. Thus $\mathcal{L}(P,M)$ is a function algebra on $M/_{\mathcal{L}(P,M)}$ and, as a closed subspace of $\mathcal{C}^\infty(M)$, also nuclear.

◘

<u>(2.4)</u> The algebras $A^\nu_{\mathbb{C}_G}$ of generalized analytic functions introduced by T. Tonev *) and generalizing the Arens-Singer concept [7], are Schwartz function algebras, provided ν is sufficiently large. It looks very likely that they are even nuclear.

Note that $j : \mathbb{C}_G \longrightarrow \sigma A^\nu_{\mathbb{C}_G}$ is homeo (i.e. a homeomorphism), hence $\sigma A^\nu_{\mathbb{C}_G}$ locally compact.
More generally, one might conjecture a positive answer to the following

<u>Problem II:</u> <u>Let</u> A <u>be</u> <u>a</u> <u>Schwartz</u> <u>function</u> <u>algebra</u> <u>with</u> <u>locally</u> <u>compact</u> σA. <u>Then</u> <u>is</u> A <u>nuclear?</u>
It is easy to give counterexamples when the local compactness assumption is dropped. For example, take a non-nuclear $(\mathcal{D}FS)$-vector space E. Then $\mathcal{O}(E)$, the algebra of all holomorphic functions on E (see (2.2)), can be shown to be a Schwartz function algebra on E without $\mathcal{O}(E)$ being nuclear.
$j : E \longrightarrow \sigma\mathcal{O}(E)$ is continuous bijective and it is easily seen that $\sigma\mathcal{O}(E)$ cannot be locally compact.

<u>(2.5)</u> Clearly, a function algebra (B,X), with X compact, never can be nuclear (except when X is a finite set).
But there are two natural constructions that often lead from a Banach function algebra to a nuclear function algebra.
Suppose $X = \sigma B$.
i) If B has a Gleason part $\Pi \subset \sigma B$ (cf. [25]) which is open and dense in σB then (B_Π, Π) happens "often" to be a nuclear function algebra.

ii) If σB is purely n-dimensional (with respect to Chevalley dimension, cf. [14]), and if $X := \sigma B - \gamma_{n-1}B$ is non-empty ($\gamma_{n-1}B$ denotes the $(n-1)^{st}$ Shilov boundary of B, introduced by Basener and Sibony, cf. [14]), then (B_X, X) is a nuclear function algebra. As a matter of fact, X even can be given the structure of a purely n-dimensional $\mathbb{C}$-analytic space s.t. B_X becomes a closed subalgebra of $\mathcal{O}(X)$; the proof is implicitly contained in [14], see also [19]. It rests upon Basener's and Sibony's theorem [2],[24].

(2.6) By using permanence properties [21] the above classes of examples often can be enlarged considerably.

a) Let (A,X) be a nuclear (resp. Schwartz) function algebra and $A_o \subset A$ a closed subalgebra.
Then $(A_o, X/_{A_o})$ is a nuclear (resp. Schwartz) function algebra.

b) Let (A_j, X_j) be nuclear (resp. Schwartz) function algebras, $1 \leq j \leq n$.
Then

$$(A_1 \times \ldots \times A_n, \bigsqcup_{\text{disjoint}} X_j) \quad \text{and}$$

$$(A_1 \hat{\otimes} \ldots \hat{\otimes} A_n, X_1 \times \ldots \times X_n)$$

are nuclear (resp. Schwartz) function algebras. ($\hat{\otimes}$ = complete slice tensor product).
The Cartesian product of algebras extends easily to the case of a countable number of factors.

c) The case of taking quotients must be handled carefully. A function algebra A is called strongly uniform if for all uniform ideals (= kernel ideals) $\check{\mathfrak{U}} \subset A$, the algebra $A/_{\check{\mathfrak{U}}}$ endowed with the natural quotient topology is a function algebra again. The Shilov boundary for Banach function algebras often turns out as an obstacle to strong uniformity. For example, the disk algebra $\mathcal{H}(\overline{\Delta})$ is not strongly uniform (see [19] for a proof communicated to me by Gamelin). But nuclear or Schwartz function algebras seem to be "often" strongly uniform.

So, whenever you have a strongly uniform nuclear (resp. Schwartz) function algebra, then you may take quotients with respect to uniform ideals and you obtain again nuclear (Schwartz) function algebras. (Note that the quotient of a Montel space need not be Montel).

Problem III : Let A be a Schwartz function algebra with locally compact σA. Then, is A strongly uniform ?

The local compactness assumption cannot be dropped.
Example: Take a $\mathcal{C}^\infty$-manifold M and put $A := \mathcal{O}((\mathcal{C}^\infty(M))')$.
By (2.2) A is a nuclear function algebra on $\mathcal{C}^\infty(M)'$. Observe that $j : M \longrightarrow \sigma\mathcal{C}^\infty(M)$ is homeo and $\sigma\mathcal{C}^\infty(M)$ is a hull for A in $\sigma A \simeq \mathcal{C}^\infty(M)'$ [13].
Thus $\check{\mathfrak{U}} := \{f \in A : f_{|\sigma\mathcal{C}^\infty(M)} = 0\}$ is a uniform ideal in A.
By Satz (2.3) in [13] (which is also valid for semi-simple nuclear F-algebras) we have

$$A_{/\check{\mathfrak{U}}} \simeq \mathcal{C}^\infty(M) \quad \text{(as top. algebras).}$$

Hence A cannot be strongly uniform, for $\mathcal{C}^\infty(M)$ is not a function algebra (see (2.7)).

A word of warning:
(2.7) Note that $(\mathcal{C}^\infty(M),M)$, M a smooth manifold, never can be a nuclear function algebra although $\mathcal{C}^\infty(M)$ is a (semi-simple) nuclear (F)-algebra of functions. For, the closure of $\mathcal{C}^\infty(M)$ with respect to the topology of compact convergence equals $\mathcal{C}(M)$ which isn't nuclear.

§ 3) Finitely generated nuclear function algebras

If, for a given function algebra A, no specific carrier space X is displayed, we always will understand σA to be the carrier space for A.
Note that none of the theorems in the next §§ can be true in the case of uniform Banach algebras (except if they are finite dimensional vector spaces).
Only a few statements will be proved in the sequel, but, in any case I'll give references to the literature. (Let me remark here that the whole complex will be worked out in great detail in the forthcoming book [19]).

(3.1) When you try to establish a structure theory for a certain class of objects you usually ought to start with the "finitely generated" objects.
It turns out that the singly generated, even Montel rather than nuclear, function algebras behave, so to say, sensationally well. But unfortunately, n-generated nuclear function algebras (n≧2) can exhibit pathologies. (Recall that a function algebra A is said to be n-generated if there are $f_1,\dots,f_n \in A$ s.t.

$$A = \overline{\mathbb{C}[f_1,\dots,f_n]}$$

but there are no n-1 elements in A doing this. The trivial algebra $\mathbb{C}$ is 0-generated).

(3.1.1) Theorem: Let $A = \overline{\mathbb{C}[f]}$ be a singly generated Montel function algebra having connected σA.
Then there is a simply connected domain $G \subset \mathbb{C}$ s.t. $A \simeq \mathcal{O}(G)$ (iso as top algebras). Moreover, G can be chosen to be $\hat{f}(\sigma A)$ and then the top algebra isomorphism is induced by $f \longrightarrow z$.

Note that the assumption "σA connected" easily can be checked by the algebra A itself rather than by its spectrum. Namely, Shilov's idempotent theorem tells us that σA is connected if and only if A has no non-trivial idempotents.

Actually, we inserted this assumption just in order to give a smoother formulation of the theorem. There is the following general version.

(3.1.2) Theorem: Let $A = \overline{\mathbb{C}[f]}$ be a singly generated Montel function algebra.
Then $\hat{f}(\sigma A) = U \cup N$ consists of at least two points, with
- $U \subset \mathbb{C}$ a simply connected open subset (possibly empty),
- $N \subset \mathbb{C} \setminus \overline{U}$ a discrete subset without clusterpoints in $\mathbb{C}$ (possibly empty),
s.t. the algebra homomorphism

$$A \longrightarrow \mathcal{O}(U) \times \mathbb{C}^N ,$$

induced by $f \longrightarrow z$, is an isomorphism of topological algebras.

Remark: Theorem (3.1.1) was proved in Goldmann [8]. During this conference the author learnt that this theorem (under the additional hypothesis "σA locally compact") was also known by D. Vogt (Wuppertal) some twelve years ago.
D. Vogt had even observed that the Montel assumption may be replaced by the weaker assumption "A reflexive".
He treated also the case of a rationally singly generated A rather than singly generated. Unfortunately, Vogt had not published his proof.
Since some further improvements had been obtained by Goldmann we decided to comprise our ideas, theorems, and proofs in a joint paper [10] which also appears in these Proc. So we need not reproduce a proof of (3.1.2) in this paper.

(3.2) As an application of thm. (3.1.1) we obtain a necessary criterion for the Schwartz property and hence for the nuclearity property.

Corollary: Let (A,X) be a Schwartz function algebra, say, on a connected X consisting of at least two points. Then, for each compact and connected $K \subset X$

there exists a larger compact $L \subset X$ *s.t. we have*

$$f(K) \subset int\,(\widehat{f(L)}_{\mathcal{P}}) \quad ,$$

for all nonconstant $f \in A_{|K}$.

(The symbol $\widehat{\ }_{\mathcal{P}}$ denotes the polynomially convex hull).

Proof: Let $f \in A$ be a non-constant function. $\overline{\mathbb{C}[f]}$, as a closed subalgebra of A, is a Schwartz function algebra. By thm. (3.1.1) it is topologically isomorphic to $\mathcal{O}(\widehat{f(X)}_{\mathcal{P}})$; in particular, $\widehat{f(X)}_{\mathcal{P}}$ is a simply connected domain in $\mathbb{C}$.

Let $\ldots \subset K_n \subset K_{n+1} \subset \ldots$ be an admissible exhaustion of X s.t. the restriction maps

$$r_n : A_{K_{n+1}} \longrightarrow A_{K_n}$$

are compact operators.

For any given compact $K \subset X$ there is $n \in \mathbb{N}$ s.t. $K \subset K_n$.

Put $L := K_{n+1}$.

Since r_n is compact, we have the same assertion for the restriction map

$$\begin{array}{ccc} \overline{\mathbb{C}[f]}_L & \longrightarrow & \overline{\mathbb{C}[f]}_K \\ \wr\Vert & & \wr\Vert \\ \mathcal{O}(\widehat{f(L)}_{\mathcal{P}}) & \rightarrow & \mathcal{O}(\widehat{f(K)}_{\mathcal{P}}). \end{array}$$

From the second line arrow one concludes as in the proof of thm.(3.1.1) that

$$\widehat{f(K)}_{\mathcal{P}} \subset int\,(\widehat{f(L)}_{\mathcal{P}})$$

as desired.

∎

One may wonder whether or not the criterion given in our Corollary might be sufficient, too. But this is far from being so.

For instance, take example (8.6) in [15]. This is a function algebra $(A,\mathbb{C}^2)$ which shares a lot of good properties with holomorphic algebras such as maximum

modulus principle, antisymmetry, and Liouville principle.
But it is just a bit more than an exercise to show (cf. [19]):
1) A is not a Schwartz l.c.s.;
2) (A,G) satisfies the assertion of the above corollary.

(3.3) We sketch the construction of a two-fold generated nuclear function algebra (A,G), $G \subset \mathbb{C}$ a domain, s.t. σA isn't even locally compact.

Hence A cannot be a Stein algebra though it is a closed subalgebra of a Riemann algebra.

Example: Choose an infinite subset $N \subset \mathbb{C}$ without cluster point in $\mathbb{C}$, $0 \notin N$, and set $G := \mathbb{C} - N$.
Next choose $f \in \mathcal{O}(\mathbb{C})$ having N as its exact zero-set V(f). Now put $A := \overline{\mathbb{C}[f, z\cdot f]}$, the closure taken with respect to compact convergence on G.
(A,G) is the desired example.

The two functions f and $z\cdot f$ separate the points of G (easy calculation!).
Since A is a closed subalgebra of $\mathcal{O}(G)$ we have that the pair (A,G) is a nuclear function algebra.

Now take an admissible exhaustion $\ldots \subset K_n \subset K_{n+1} \subset \ldots$ of G by compact subsets.
By a standard function algebra argument (cf. e.g. [15], (7.6)) we have that

$$(f,\ z\cdot f) : \sigma A \longrightarrow \mathbb{C}^2$$

maps σA continuously and bijectively onto the set $Y := \bigcup_{n\in\mathbb{N}} Y_n$ with

$$Y_n := \widehat{(f,\ z\cdot f)(K_n)}_{\mathcal{P}} \subset \mathbb{C}^2 \quad .$$

Y contains the point (0,0) which does not have a compact nbd in Y. Note that $\ldots \subset Y_n \subset Y_{n+1} \subset \ldots$ is not an admissible exhaustion of Y with respect to the Euclidean topology of $Y \subset \mathbb{C}^2$. For the details and further observations of this construction we refer the reader to [19].

We just mention one more application of this example. It can be reformulated in such a way to yield a negative answer to the following
Problem: Let $G \subset \mathbb{C}^n$ be a domain of holomorphy and $M \subset G$ be either a 1-dimensional or a 1-codimensional complex (hence Stein) submanifold. It is well-known that $\hat{G}_{\mathcal{P}}$ is again a domain of holomorphy. But is $\hat{M}_{\mathcal{P}} \subset \hat{G}_{\mathcal{P}}$ a submanifold (or, at least, a complex subspace) again?

In our example, $\hat{M}_{\mathfrak{P}}$ isn't even locally compact.
Our example leads to the outstanding

Problem IV: Let A be a finitely generated nuclear function algebra with locally compact σA. Then, is A a Stein algebra?

There are positive results in some particular cases, which are sketched in [9].

§ 4 Some properties of nuclear function algebras

(4.1) Let's begin with a simple observation:
Let A be a Schwartz function algebra. If σA is compact then it consists of only a finite number of points and A is a finite dimensional vector space. (For A were Schwartz and Banach at the same time!).

As a matter of fact, a bit more work yields the same assertion even for Montel function algebras rather than for Schwartz ones. Namely, a Banach function algebra never can be reflexive except in the trivial case of being a finite dimensional vector space (see [19]).
A consequence of this fact is that any compact analytic set in a holomorphically separable complex space (e.g. in a Stein space) consists of only a finite number of points.

(4.2) The Shilov boundary of a Schwartz function algebra A is empty; A does not even posses independent points in the sense of Rickart [23]. Heuristically speaking, the Schwartz property pushes the Shilov boundary out to infinity (like in the constructions (2.5)). This fact implies the

Proposition: If σA is locally compact then A is a Max-Mod algebra on σA, i.e.

$$\| f\|_K = \| f\|_{\partial K} \quad \textit{for all } f \in A$$

and for all compact $K \subset \sigma A$ *with* $\overset{\circ}{\overline{K}} = K$.

For a proof cf. [12],[20].

The concept of the Basener Sibony Shilov-boundaries of higher order can be made meaningful in this setting and yield a fruitful interpretation concerning the dimension of the spectrum ([19]).

(4.3) We wish to use Proposition (4.2) in order to meditate a bit on the existence of nuclear function algebras on hemi-compact spaces of real dimension 1.

(4.3.1) Proposition: There is no nuclear function algebra A with σA homeomorphic to $\mathbb{R}$. In fact, the nuclearity assumption may even be weakened to "Montel".

Proof: By [20] we knowe that any Montel function algebra with locally compact and connected spectrum is a maximum modulus principle algebra. We may assume $\sigma A = \mathbb{R}$ and exhaust $\mathbb{R}$ by
$K_n := [-n,n]$. Hence we have

$$A_{K_n} = A_{\partial K_n} = A_{\{\pm n\}} .$$

But a function algebra on two singletons is necessarily isomorphic to $\mathbb{C}^2$. Thus

$$A = \lim_{\longleftarrow} A_{K_n} = \lim_{\longleftarrow} \mathbb{C}^2 = \mathbb{C}^2 .$$

Contradiction !

∎

(4.3.2) Proposition: There exist one-dimensional $\mathbb{R}$-analytic spaces X which admit nuclear function algebras (A,X).
It can be shown that such examples never can be locally nuclear (see (4.8)).
For the proof take

$$X := \mathbb{R} \cup \bigcup_{n=1}^{\infty} \partial\Delta_n \subset \mathbb{C} ,$$

with $\Delta_n := \{z \in \mathbb{C} : |z| < n\}$.

Obviously X is a one-dimensional $\mathbb{R}$-analytic space. Now put

$$A := \mathcal{O}(\mathbb{C})_{|X} .$$

Clearly, the restriction

$$\mathcal{O}(\mathbb{C}) \longrightarrow \mathcal{O}(\mathbb{C})_{|X}$$

is a topological isomorphism, hence (A,X) a nuclear function algebra.
For sufficiently small open $U \subset X$ you see immediately (e.g. by the Stone-Weierstraß theorem) that

$$A_U = \mathscr{C}(U) .$$

Thus A is not locally nuclear. (The general case is more involved).

Problem V: Does there exist a nuclear function algebra (A,X) with X homeomorphic to $\mathbb{R}$?

(Note that by (4.2) the case $X = \sigma A = \mathbb{R}$ is impossible !)

(4.4) For a Schwartz (even Montel) function algebra Gelfand topology and strong topology on σA are compactologically equivalent [13], hence identical if σA is a k-space w.r. to the Gelfand topology.
(In comparison, look at the Banach algebra case: σB is compact in the Gelfand topology, but σB is never compact in the strong = metric topology ($\dim B = \infty$); e.g. peak points for B in σB become singletons in the strong topology.)
This fact admits some applications. Basically,

1) function algebra reasoning usually is done w.r. to the Gelfand topology, whereas
2) doing function theory in A' is meaningful only w.r. to the strong topology (note that $\sigma A \subset A'$ can be interpreted as an analytic set, [19]).

Hence by the above fact, these two different working perspectives live on the uniquely topologizable carrier space σA.
A further little consequence from the above may be drawn for the function algebra $A_{\mathbb{C}_G}$ of Arens-Singer functions ($= A^{\nu}_{\mathbb{C}_G}$ with $\nu=0$, see (2.4)), whenever G is the dual group of a dense, discrete subgroup of $(\mathbb{R},+)$. Instead of taking $A_{\mathbb{C}_G}$ you equally well may consider the original Arens-Singer algebra A_G, and concentrate on a sufficiently small connected nbd U of $*$ of σA_G, the vertex of the cone.
$\mathbb{C}_G$ and U are connected w.r. to the Gelfand topology. But not so w.r. to the strong topology since $*$ is a one point Gleason part and hence a singleton (see [7], chap VII).
Thus neither $A_{\mathbb{C}_G}$ nor $(A_G)_U$ are Schwartz (not even Montel) function algebras!

(4.5) Proposition: Any Montel function algebra is antisymmetric.
Recall that A is said to be antisymmetric if $f \in A$ and $\overline{f} \in A$ ($\overline{f}$ = complex conjugate) implies f = const.

Proof: It's no loss of generality to assume that σA is connected.

Now suppose there is a non-constant $f \in A$ s.t. $\overline{f} \in A$, too. Consider the closed subalgebra $\overline{\mathbb{C}[f]}$ of A which is also Montel. By thm. (3.1.1) there is a simply connected domain $G \subset \mathbb{C}$ s.t.

$$\overline{\mathbb{C}[f]} \simeq \mathcal{O}(G) \quad \text{(top alg. iso)}.$$

The Stone-Weierstraß thm. now yields

$$\overline{\mathbb{C}[f,\overline{f}]} \simeq \mathcal{C}(G) \quad \text{(top alg. iso)}.$$

But $\mathcal{C}(G)$ is far from being Montel.
Thus we found the closed non-Montel subalgebra $\overline{\mathbb{C}[f,\overline{f}]}$ of A, contradicting its Montel property !
Hence such an f cannot exist.

□

(4.6) Like in complex analysis of several variables, there is a principle of semicontinuity of fibre dimensions for mappings

$$(\hat{f}_1,\ldots,\hat{f}_n) : \sigma A \longrightarrow \mathbb{C}^n, \; f_j \in A,$$

more precisely, for each $\varphi \in \sigma A$ there is a nbd $U \subset \sigma A$ of φ s.t.

$$\dim_\psi \hat{f}^{-1}(\hat{f}(\psi)) \leq \dim_\varphi \hat{f}^{-1}(\hat{f}(\varphi)), \quad \forall \psi \in U.$$

Here, A is supposed to be a strongly uniform (s.(2.6)) Schwartz function algebra having locally compact σA. A proof is found in [12].

In complex analysis this theorem actually rests upon the Weierstraß theorem. Note that in our setting this local theory is not at all available. So we had to develop completely new and independent proofs.

(4.7) Now we quote a nice characterization of nuclearity for function algebras on "good" carrier spaces. This goes in terms of the existence of certain volume integral formulae. The criterion is due to Pietsch [22]; as a matter of fact, he proved it, more generally, for function spaces. (A function space is a thing exactly as in (1.1) when you replace the word "algebra" by "space" (meaning "vector space")).

Theorem: Let (A,X) be a function algebra. Assume that X has a countable *basis for its topology.* (*)

Then A is nuclear if and only if the following holds:

1) *there exists a positive Radon measure* μ *on* X;

2) *for each compact* $K \subset X$ *there exists a larger compact* $L \subset X$ *and a* μ*-measurable and* μ*-bounded function* $k : K \times L \longrightarrow \mathbb{C}$, *s.t.*

3) $f(x) = \int_L f(y)k(x,y)d\mu(y), \quad \forall\, x \in K, \quad \forall\, f \in A$.

We do not reproduce the proof here. Note that the integral representation 3) immediately yields the Schwartz property of A, for the restriction map $r : A_L \longrightarrow A_K$ given by the integral is compact. A bit more analysis shows that r is even nuclear.

By the algebra property of A we have for all $f,g \in A_L$ and all $x \in K$:

$$\int_L (f\cdot g)(y)k(x,y)d\mu(y) = \int_L f(y)k(x,y)d\mu(y)\cdot\int_L g(y)k(x,y)d\mu(y).$$

So, in actual situations, the quality of the kernel function k often can be improved considerably.

(4.8)
Definition: A function algebra (A,X) is called locally nuclear (resp. locally Schwartz) if there is a basis $\mathcal{B}$ *for the topology of X (consisting of hemi-compact open subsets) s.t. the localizations* A_U *are nuclear (resp. Schwartz), for all* $U \in \mathcal{B}$.

In many instances the hemi-compactness condition is automatically satisfied, e.g. when A is separable and X is locally compact .
This notion intensifies the notion of nuclearity (resp. Schwartzity) :

(4.8.1) Lemma: If (A,X) is a locally nuclear (resp. locally Schwartz) function algebra then it is nuclear (resp. Schwartz).

Proof: Consider the algebra

$$A_1 = \prod_{U\in\mathcal{B}} A_U \quad .$$

This is a nuclear (resp. Schwartz) l.c.s. with respect to the product topology. Now observe that the mapping

$$A \longrightarrow A_1, \quad f \longrightarrow (f_{|U})_{U\in\mathcal{B}},$$

(*) Note that this countability condition for the topology of X implies the local compactness of X. Cf. Goldmann-Kramm-Vogt [10], Lemma (4.8).

is an embedding of A as a closed subspace of A_1. Hence A is a nuclear (resp. Schwartz) l.c.s. .

□

An useful reformulation of our notion is given by

(4.8.2) Proposition: A function algebra (A,X) is locally nuclear (resp. locally Schwartz) iff for each compact $K \subset X$ and each (hemi-compact) nbd U of K in X there is a larger compact $L \subset U$ s.t. the restriction map

$$A_L \longrightarrow A_K \text{ is nuclear (resp. compact) .}$$

Moreover, if X is locally compact then this is equivalent to the fact that the restriction map $A_L \longrightarrow A_K$ is nuclear (resp. compact) whenever $K \subset L^o$.

For a proof use the argument given in (4.8.1) and apply it to (A_U,U) .

□

(4.8.3) All the standard examples (2.1),(2.2),(2.3) are also locally nuclear. It's very likely that the examples in (2.4) are locally Schwartz (may be even locally nuclear), if ν is large enough.
I'm aware of no case $(A,\sigma A)$ where this is not so. So let's formulate this as our
Problem VI: Let A be a nuclear (resp. Schwartz) function algebra. Then, is A necessarily locally nuclear (resp. locally Schwartz) on σA ?

It is easy to give negative examples (A,X) with $X \neq \sigma A$; e.g. take example (4.3.2): for sufficiently small open $U \subset X$ you obtain $A_U = \mathcal{C}(U)$ which clearly isn't Schwartz. We collect some rather strong properties of locally nuclear function algebras in the next two theorems.

(4.8.4) Theorem: Let (A,X) be a locally Schwartz function algebra on a locally compact X. Then:

1) *For each $x \in X$ there exists a nbd $U \subset X$ of x s.t. for all nbds $W \subset U$ of x and all $f \in A$ we have:*
   $f_{|W} \equiv 0 \Rightarrow f_{|U} \equiv 0.$
   (Weak identity theorem).
2) *For $X = \sigma A$ we have: σA is locally connected.*

Proof:

1) A is separable since it's Schwartz. From this and the local compactness assumption on X one can conclude that X is first countable.
So, for our given $x \in X$ choose a countable nbd basis

$$\dots \supset U_n \supset U_{n+1} \supset \dots$$

of open subsets satisfying $\overline{U}_{n+1} \subset U_n$.

Hence the restrictions $A_{\overline{U}_n} \longrightarrow A_{\overline{U}_{n+1}}$ are compact operators.

This yields the strict $\underrightarrow{\lim}$ topology on

$$A_x := \underset{\overrightarrow{n}}{\lim} A_{\overline{U}_n}$$

which therefore is Hausdorff [5].

Now introduce the ideals $\mathfrak{A}_n := \{f \in A : f_{|U_n} \equiv 0\}$ and

$\mathfrak{A} := \{f \in A : (f)_x \in A_x$ is the zero-germ$\}$.

Trivially the $\mathfrak{A}_n$ are closed; $\mathfrak{A}$ is closed, too, since

α) the ideal $\{(0)_x\}$ is closed (Hausdorff property of A_x);

β) the map $A \longrightarrow A_x$, $f \longrightarrow (f)_x$, is continuous.

Obviously we have

$$\dots \subset \mathfrak{A}_n \subset \mathfrak{A}_{n+1} \subset \dots \subset \mathfrak{A} = \bigcup_n \mathfrak{A}_n .$$

Since all these ideals are F-spaces we may apply Baire's theorem.
Hence there is an index $n_0 \in \mathbb{N}$ s.t.

$$\mathfrak{A}_{n_0} = \mathfrak{A} .$$

$U := U_{n_0}$ is the nbd of x with the desired properties.

2) Thm. 6 in [17].

(4.8.5) By $\mathcal{M}(K)$, $K \subset X$ compact, we denote the Banach space of complex Radon measures on K.

Theorem (Existence of reproducing boundary value integrals):
Let A be a locally nuclear function algebra on the locally compact spectrum σA.
Let $U \subset \sigma A$ be an open relatively compact subset.
Then there exists an A_U-morphic map

$$\mu : U \longrightarrow \mathcal{M}(\partial U),\ x \longrightarrow \mu_x \ \text{s.t.}$$

$$f(x) = \int_{\partial U} f(y)\, d\mu_x(y),\ \forall f \in A_{\overline{U}} \ \text{and} \ \forall x \in U .$$

"A_U-morphic" means:

1) μ is continuous;
2) for all $\psi \in \mathcal{M}(\partial U)'$ we have:

$$\psi \circ \mu \in A_U \quad .$$

This notion generalizes holomorphic measure-valued maps to our setting. The integral formula of our theorem may be regarded as a general Cauchy-Weil integral formula. For the proof we refer the reader to [17].

Regrettably, at this time the author has fallen sick very seriously.
So the paper could not be finished. The reader is referred to the forthcoming book [19] .

NOTE.

B. Kramm died on October 11, 1983. See the editorial comment in the preface.

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Functional Analysis: Surveys and Recent Results III
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REFLEXIVE FUNCTION ALGEBRAS

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Introduction

In the present note we study singly generated and singly rationally generated Fréchet function algebras (A,X) on a hemicompact k-space (resp. locally compact σ-compact space) X, which are reflexive as topological vector spaces. By use of a decomposition with respect to their Gleason parts we can reduce the investigation to the case of a connected spectrum σA. We show that in this case A is either trivial or isomorphic to the algebra $\mathcal{O}(U)$ of holomorphic functions on a connected open subset $U \subset \mathbb{C}$ which in the singly generated case can be assumed to be either $\mathbb{C}$ or $D := \{z \in \mathbb{C} : |z| < 1\}$.

This gives on one hand a classification of these algebras, on the other hand it can be considered as a characterization of algebras of type $\mathcal{O}(U)$. There are various other characterizations of $\mathcal{O}(U)$. Rudin [1] (resp. Meyers [1]) proved, that for a function algebra (A,U), containing the polynomials and with spectrum U, the following conditions are equivalent:

i) $A = \mathcal{O}(U)$

ii) A is a maximum modulus algebra

iii) A is a Montel algebra.

Arens [1] showed that a function algebra which is rationally singly generated by z and which has a continuous derivation D such that Dz = 1 and such that

$$\|D^k f\|_{K_n} \le k!\, r_n^{-k} \|f\|_{K_{n+1}}$$

where $r_0, r_1, \ldots$ is a sequence of positive reals, must be topologically and algebraically isomorphic to $\mathcal{O}(U)$ with some open subset $U \subset \mathbb{C}$. In Theorem (4.1.) we also characterize algebras of type $\mathcal{O}(U)$ as those singly rationally generated function algebras which have locally compact (and connected) spectrum and satisfy the maximum modulus principle.

In the rest of the paper we are concerned with properties of reflexive algebras (resp. maximum modulus algebras). It is well known that Gleason parts are important to answer the question whether there exists analytic structure in the spectrum. In section 2 we prove that reflexive function algebras have "big" Gleason parts, i.e. if $(A,\sigma A)$ is reflexive then the Gleason parts are exactly the connected components of σA.

In section 5 we examine the range of elements of a reflexive function algebra. In particular we show that if $(A,\sigma A)$ is a reflexive function algebra with connected spectrum and $f \in A$ is a non constant function, then $f(\sigma A)$ has positive plane Lebesgue measure. In section 6 we prove a weak identity theorem for certain points $x \in X$, where (A,X) is a maximum modulus algebra.

1. Preliminaries

(1.1) For basic concepts and notations we refer to the survey article [3] of B. Kramm in this same volume. A function algebra is a closed separating subalgebra of the Fréchet algebra $C(X)$ of continuous functions on a hemicompact k-space X (e.g. a locally compact σ-compact space X). $C(X)$ is always equipped with the c.o.-topology. A fundamental system of seminorms is given by $\|f\|_n = \sup\{|f(x)| : x \in K_n\}, n=1,2,\ldots$ where $(K_n)_n$ is an admissible exhaustion of X, i.e. $K_n \subset K_{n+1}$, K_n compact for every n and for every compact $K \subset X$ there is n with $K \subset K_n$. The set σA of all continuous multiplicative functionals on A equipped with the weak*-topology is called the spectrum of A. If K is a compact set in the plane we consider the following algebras:

$P(K) := \{f \in C(K) : f$ can be approximated uniformly on K by polynomials$\}$,
$R(K) := \{f \in C(K) : f$ can be approximated uniformly on K by rational functions with poles off $K\}$.

We state some well known facts (see for instance Stout [1] or Gamelin [1]). The spectrum $\sigma R(K)$ of $R(K)$ can be identified with K, $\sigma P(K)$ can be naturally identified with the polynomially convex hull $\hat{K}$. The topological boundary of K is the Shilov boundary for $R(K)$, and the Shilov boundary for $P(K)$ is the topological boundary of $\hat{K}$. Moreover every point of $\gamma P(K)$ (Shilov boundary for $P(K)$) is a peak point for $P(K)$, i.e. for every $x \in \gamma P(K)$ there exists $f \in P(K)$ with $f(x) = 1$ and $|f(y)| < 1$ for all $y \in K\setminus\{x\}$. In general the same is not true for $R(K)$, but of course we have $\gamma R(K) = \overline{\chi R(K)}$ where $\chi R(K) := \{x \in K : x$ is peak point for $R(K)\}$.

(1.2) Let $(A,\sigma A)$ be a singly rationally generated function algebra with generating element f, $\tilde{K}_n \subset \tilde{K}_{n+1} \subset \ldots$ be an admissible A-convex exhaustion of σA, then $f : \tilde{K}_n \to f(\tilde{K}_n) =: K_n$ is a homeomorphism, $A_{\tilde{K}_n}$ (the restriction algebra $A_{|\tilde{K}_n}$ completed in the norm $\|\cdot\|_{\tilde{K}_n}$) is isomorphic to $R(K_n)$ and A is topologically and algebraically isomorphic to $\varprojlim R(K_n)$, the projective limit with respect to the natural restriction mappings. If $(A,\sigma A)$ is singly generated we have in addition that K_n is polynomially convex for all $n \in \mathbb{N}$ and therefore $R(K_n) = P(K_n)$.

(1.3) In this paper we consider reflexive function algebras, i.e. function algebras for which $A''=A$ or equivalently and more relevant for our work: every bounded subset is weakly relatively compact. Note that every Montel function algebra is reflexive. A function algebra (A,X) is said to be Montel if it is a Montel space as a topological linear space (i.e. if every bounded subset of A is relatively compact in A).

2. Gleason parts for reflexive function algebras

In [2] , Thm. 7.2 Kramm shows the following: Let A be a strongly nuclear function algebra with locally compact spectrum σA, such that all open subsets of σA are hemicompact. Then the Gleason parts for A are exactly the connected components of σA (which are all open).

We shall prove the same result for reflexive function algebras (A,X).

(2.1) Definition: *Let* (A,X) *be a function algebra and* $K_n \subset K_{n+1} \subset \dots$ *an admissible exhaustion of* X. *We say that* $x,y \in X$ *belong to the same Gleason part* $(x \sim y)$ *if there exists* $n \in \mathbb{N}$ *such that* x *and* y *belong to the same Gleason part of* A_{K_n} $(x \underset{n}{\sim} y)$.

Recall that $x \underset{n}{\sim} y$ if $\sup \{|f(x)-f(y)| : f \in A, \|f\|_{K_n} \le 1\} < 2$, which is equivalent to $\sup \{|f(x)| : \|f\|_{K_n} \le 1, f(y) = 0, f \in A\} < 1$ (see Stout [1] (16.1)).

Remark: "$\sim$" is an equivalence relation. The transitivity is easily seen, namely from $x \underset{n}{\sim} y$ follows $x \underset{n+p}{\sim} y$ for all $p \in \mathbb{N}$.

(2.2) Theorem: *Let* (A,X) *be a reflexive function algebra, then the Gleason parts for* A *are open.*

Proof: We argue indirectly. Suppose there exists a point $x \in X$ such that the equivalence class $[x] := \{y \in X : x \sim y\}$ is not an open subset of X. Then in every neighbourhood U of X there is a $y_U \in U$ with $y_U \notin [x]$ which means $y_U \underset{n}{\not\sim} x$ for all $n \in \mathbb{N}$. Therefore for each U of the neighbourhood filter $\mathcal{U}$ of x and each n we find a function $f_n^U \in A$ with $\|f_n^U\|_{K_n} \le 1$, $f_n^U(y_U) = 0$ and $f_n^U(x) \ge 1 - \frac{1}{n}$. Since A is reflexive and the sequence $(f_n^U)_n$ is bounded we get a function $f^U \in A$ with $\|f^U\|_{K_n} \le 1$ for all $n \in \mathbb{N}$, $f^U(y_U) = 0$ and $f^U(x) = 1$.

Now for every finite subset $M \subset \mathcal{U}$ we define the function $g_M := \prod_{U \in M} f^U$. The g_M, where the M's are directed by inclusion, are a bounded net in the reflexive space A which therefore has a subnet converging to some $g \in A$ (Kelley [1] , p. 136). Since $g(y_U) = 0$ for all $U \in \mathcal{U}$ and $g(x) = 1$ the function g is not continuous. This is a contradiction.

(2.3) Corollary: *If* $(A,\sigma A)$ *is a reflexive function algebra, then the Gleason parts are exactly the connected components of* σA.

Proof: In (2.2) we showed that for every $x \in \sigma A$ the set $[x]$ is clopen. Assume $[x]$ is not connected, then there is a clopen nonempty proper subset $M \subset [x]$. By Shilov's idempotent theorem, which is also valid for function algebras (see Kramm [1], (7.2)), χ_M lies in A. This is a contradiction.

3. The maximum modulus principle for reflexive function algebras

Let (A,X) be a function algebra.

(3.1) Definition: (A,X) *satisfies the maximum modulus principle, if for every compact subset* $K \subset X$ *without isolated points the Shilov boundary of* A_K *is contained in the topological boundary of* K.

(3.2) Definition: *A compact subset* $S \subset X$ *is said to be a local peak set of* A, *if there is a neighbourhood* U *of* S *and an element* $f \in A$ *such that* $f(x) = 1$ *for* $x \in S$ *and* $|f(x)| < 1$ *for* $x \in U \setminus S$.

Meyers [1] proved the following results (Proposition 1 and Corollary 1) for Montel algebras. His proofs need only reflexivity, i.e. the fact that bounded sets are relatively weakly compact.

(3.3) *Let* (A,X) *be a function algebra on a locally compact space* X. *If* A *is reflexive, then every local peak set of* A *in* σA *is clopen in* σA.

(3.4) *Let* (A,X) *be a function algebra on a locally compact and connected space. If* A *is reflexive then* A *is a maximum modulus algebra on* σA.

4. A characterisation theorem for singly rationally generated reflexive (resp. maximum modulus) function algebras

(4.1) Theorem: *Let* $(A,\sigma A)$ *be a singly rationally generated maximum modulus algebra with connected and locally compact spectrum* σA *and generating element* f.

Then either $A \cong \mathbb{C}$ *or* $f(\sigma A)$ *is an open subset of the complex plane and* A *is topologically and algebraically isomorphic to* $\mathcal{O}(f(\sigma A))$, *the holomorphic functions on* $f(\sigma A)$.

Proof: If $A \neq \mathbb{C}$ then σA contains at least two points. Let $\tilde{K}_n \subset \tilde{K}_{n+1} \subset \dots$ be an admissible A-convex exhaustion of σA, then A is topologically and algebraically isomorphic to $\varprojlim R(K_n)$, with $K_n := f(\tilde{K}_n)$.

Now let φ be an arbitrary point of σA. There exists $n_0 \in \mathbb{N}$ such that $\tilde{K}_{n_0}$ is a compact neighbourhood of φ. $(A,\sigma A)$ is a maximum modulus algebra, therefore $\varphi \notin \gamma A_{\tilde{K}_{n_0}}$.

From Gamelin [1] p. 1o we learn that the topological boundary ∂K_{n_0} is contained in $f(\gamma A_{\tilde{K}_{n_0}})$, f is injective, hence $f(\varphi)$ is an interior point of K_{n_0} and we have $\bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} \mathring{K}_n$.

Remark: If σA is not connected, A is topologically and algebraically isomorphic to $\mathcal{O}(G) \times C(N)$, with an open subset $G \subset \mathbb{C}$ (possibly empty) and an at most countable subset $N \subset \mathbb{C}$ (possibly empty), which has no cluster point and satisfies $\bar{G} \cap N = \emptyset$.

(4.2) Corollary: *If* $(A,\sigma A)$ *- under the assumptions of* (4.1) *- is singly generated, then* A *is topologically and algebraically isomorphic to either* $\mathbb{C}$ *or* $\mathcal{O}(D)$ *or* $\mathcal{O}(\mathbb{C})$.

Proof: If A is singly generated and nontrivial, then every K_n (defined as in the proof above) is polynomially convex and $f(\sigma A)$ is therefore a polynomially convex open set. By application of the Riemann mapping theorem one completes the proof.

(4.3) The following example shows that the assumption of local compactness in (4.2) and (4.1) is necessary: For all $n \in \mathbb{N}$ we put $K_n := \{z \in \mathbb{C} : |z| \leq 1 - \frac{1}{n}\} \cup \{z \in \mathbb{R} : 0 \leq z \leq 1\}$. Then all compacta K_n are polynomially convex. We consider the algebra $A := \varprojlim P(K_n)$.

We have $A_{K_n} = P(K_n)$, for all $n \in \mathbb{N}$, thus $\sigma A = \bigcup_n \sigma A_{K_n} = \bigcup_n K_n = D \cup \{1\}$, because all K_n are A-convex, and $K_n \subset K_{n+1} \subset \ldots$ is an admissible exhaustion of σA (see Kramm [1], (6.8)).

The Gelfand topology on $D \cup \{1\}$ is finer than the induced euclidean topology (for every $y \in \sigma A$ sets of the form

$N_{f_1,\ldots,f_n,\varepsilon,y} := \{x \in D \cup \{1\} : |f_i(y) - f_i(x)| < \varepsilon \text{ , } i=1,\ldots,n\}$,

$f_1, \ldots, f_n \in A$, $\varepsilon > 0$ are a basis of open neighbourhoods of y), hence every compact subset K of σA is also a compact subset of the plane. Moreover on K both topologies are equivalent, therefore σA is connected.

If f is an arbitrary element of A, then $f_{|D}$ is holomorphic on D and it follows immediately that A is a maximum modulus algebra.

The identity map $f(z) = z$ generates A but $f(D \cup \{1\})$ is not open. It is easy to show that the point $1 \in \sigma A$ has no compact neighbourhood.

From (3.4) we learn that (4.1) and (4.2) are also valid for reflexive function algebras. Moreover we show that for singly generated reflexive function algebras the assumption of local compactness can be dropped.

(4.4) Lemma: *If* $(A,\sigma A)$ *is a singly generated reflexive function algebra, then* σA *is locally compact*.

Proof: We use the notations of the proofs above. Assume there is a $\varphi \in \sigma A$ without compact neighbourhood. We may assume $\varphi \in \tilde{K}_1$, then we have $x := f(\varphi) \in \partial K_n = \chi P(K_n)$ for all $n \in \mathbb{N}$.
Now we choose corresponding peak functions $\tilde{f}_n \in P(K_n)$ with $\|\tilde{f}_n\|_{K_n \setminus U_n} \leq \frac{1}{2n}$, for all $n \in \mathbb{N}$, where $U_n \supset U_{n+1} \supset \ldots$ is an open neighbourhoodbasis of x in $\mathbb{C}$.
By approximating the peak functions, we get a sequence $(p_n)_n$ of polynomials with

1) $\|p_n\|_{K_n} \leq \frac{5}{4}$

2) $p_n(x) = 1$

3) $\|p_n\|_{K_n \setminus U_n} \leq \frac{1}{n}$.

The sequence $(p_n \circ f)_n$ is bounded in A and converges pointwise to the characteristic function $\chi_{\{\varphi\}}$. Because A is reflexive $\chi_{\{\varphi\}}$ is in A and φ is therefore an isolated point of σA. This contradicts our assumption on φ.

We reformulate (4.2) for reflexive function algebras .

(4.5) Theorem: *Let* $(A,\sigma A)$ *be a singly generated reflexive function algebra with connected spectrum* σA *and* $A \neq \mathbb{C}$. *Then* A *is topologically and algebraically isomorphic either to* $\mathcal{O}(D)$ *or to* $\mathcal{O}(\mathbb{C})$.

We do not know if an analogue to (4.4) holds in the singly rationally generated case. In this case K_n is not necessarily polynomially convex and ∂K_n need not consist only of peak points of R(K). Hence we need information on the behaviour of the Shilov boundaries $\gamma A_{\tilde{K}_n}$. $(\tilde{K}_n)_n$ now denotes an admissible exhaustion of X.

(4.6) Lemma: *Let* A *be reflexive,* $A \neq \mathbb{C}$ *and* σA *connected, let* X *satisfy the first axiom of countability, then* $\liminf_n \gamma A_{\tilde{K}_n} = \emptyset$.

Proof: We have to prove that $\bigcap_{n \geq n_0} \gamma A_{\tilde{K}_n} = \emptyset$ for all n_0. Assume there is

$x \in \bigcap_{n \geq n_o} \gamma A_{\tilde{K}_n}$. Let U_m, $m = 1,2, \ldots$ be a base for the neighbourhood system of X. For every $n \geq n_o$ there is $g_n \in A$ with $\|g_n\|_n := \|g_n\|_{\tilde{K}_n} = 1$ and

(1) $$|g_n(y)| \leq 2^{-n} \quad \text{for } y \in K_n \setminus U_n \quad .$$

We distinguish two cases: First we assume there is a subsequence $(g_{n_k})_k$ such that $|g_{n_k}(x)| = 1$ for all k. Then the sequence $\tilde{g}_k := g_{n_k} / g_{n_k}(x)$, $k = 1,2, \ldots$ is bounded in A and converges pointwise to the characteristic function $\chi_{\{x\}}$. Since A is reflexive this implies $\chi_{\{x\}} \in A$, which contradicts our assumption.

In the other case we have n_1 such that $|g_n(x)| < 1$ for all $n \geq n_1$. We may assume $n_1 \geq 3$ and $0 \leq g_n(x) \leq 4^{-n}$ for $n \geq n_1$. Starting with n_1 we determine inductively a subsequence. Let $n_1, \ldots, n_k$ be chosen. We choose $n_{k+1} > n_k$ so large that

(2) $$U_{n_{k+1}} \cap \bigcup_{i=1}^{k} \{y : |g_{n_i}(y)| > 2^{-n_i}\} = \emptyset \ .$$

We put

$$f_m = \sum_{k=1}^{m} g_{n_k} \quad .$$

For any k there is $x_{n_k} \in \tilde{K}_{n_k} \cap U_{n_k}$ such that $|g_{n_k}(x_{n_k})| = 1$. We obtain for $j \leq m$

$$|f_m(x_{n_j})| \geq 1 - \sum_{k=1}^{j-1} |g_{n_k}(x_{n_j})| - \sum_{k=j+1}^{m} |g_{n_k}(x_{n_j})|$$

$$\geq 1 - \sum_{k=1}^{j-1} 2^{-n_k} - \sum_{k=j+1}^{m} 2^{-n_k} \geq \frac{3}{4} \quad .$$

The second inequality follows from (1) and (2), the third from $n_1 \geq 3$.

Moreover we have for all m

$$|f_m(x)| \leq \sum_{k=1}^{m} 4^{-n_k} \leq \frac{1}{3} \quad .$$

For $y \in K_{n_j}$ and all m we have

$$|f_m(y)| \leq \sum_{k=1}^{j} |g_{n_k}(y)| + \sum_{k=j+1}^{m} |g_{n_k}(y)| \leq \sum_{k=1}^{j} \|g_{n_k}\|_{n_j} + 2 \quad ,$$

because in the second sum there is at most one term greater than 2^{-n_k} and this one is less or equal to 1.

Consequently the sequence $(f_m)_m$ is bounded in A, hence it has a weak cluster point $f \in A$. For f we have: $|f(x_{n_j})| \geq 3/4$ for all j, $|f(x)| \leq 1/3$. Since $(x_{n_j})_j$ converges to x this contradicts the continuity of f.

For $X = \sigma A$ the following theorem is an immediate consequence of (3.4) and (4.1).

(4.7) Theorem: *Let* (A,X) *be a singly rationally generated reflexive function algebra with connected spectrum* σA *on a locally compact space* X. *Let* f *be a generating element. Then either* $A = \mathbb{C}$ *or* $f(\sigma(A))$ *is an open subset of the complex plane and* A *is topologically and algebraically isomorphic to* $\mathcal{O}(f(\sigma A))$.

Proof: Since A is separating, f is injective, hence X satisfies the first axiom of countability. Because of (4.6) we may assume the $\tilde{K}_n$ chosen in such a way that $\gamma A_{\tilde{K}_n} \cap \gamma A_{\tilde{K}_{n+1}} = \emptyset$. This implies (cf. the proof of (4.1)) that $\partial K_n \cap \partial K_{n+1} \subset f\gamma A_{\tilde{K}_n} \cap f\gamma A_{\tilde{K}_{n+1}} = \emptyset$, where $K_n = f(\tilde{K}_n)$. Hence $K_n \subset \overset{o}{K}_{n+1}$.

Clearly (4.7) implies that A has locally compact spectrum. To compare the assumptions in (4.6) and (4.7) it is useful to have

(4.8) Lemma: *If* X *is hemicompact and satisfies the first axiom of countability then it is locally compact.*

Proof: Let $K_n \subset K_{n+1} \subset \ldots$ be an admissible exhaustion of X. Assume there exists a point $x \in X$ without compact neighbourhood. Let $U_n \supset U_{n+1} \supset \ldots$ be a countable basis for the neighbourhoods of x. Now choose for every $n \in \mathbb{N}$ a point $x_n \in U_n \setminus K_n$, then $K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact, but there is no $n \in \mathbb{N}$ with $K \subset K_n$. This is a contradiction.

(4.9) In (4.2) we showed that every singly generated maximum modulus algebra with locally compact spectrum is a Stein algebra and has for that reason analytic structure at any point of its spectrum. Now it is natural to ask whether the same is true for finitely generated maximum modulus algebras. We shall give a counterexample even in the doubly generated case.

Stolzenberg [1] constructed a compact subset X of the topological boundary of the unit bicylinder B in $\mathbb{C}^2$ such that the polynomially convex hull $\hat{X}$ does not carry analytic structure at any point of $\hat{X}$ and $\hat{X} \setminus X$ is not empty.

For every $n \in \mathbb{N}$ define $K_n := \hat{X} \cap \{z \in \mathbb{C}^2 : |z| \leq 1 - \frac{1}{n}\}$ and consider the algebra $A := \varprojlim P(K_n)$ where $P(K_n) := \{f \in C(K_n) :$ f can be approximated uniformly on K_n by polynomials$\}$. Every K_n is polynomially convex and it is easy to check that $\sigma A = \hat{X} \setminus \partial B$ and that the Gelfand topology and the induced euclidean topology are equivalent on σA. Therefore σA is connected and locally compact.

Now let U be a relative compact open subset of σA, then U is also a relative compact open subset of $\hat{X}$ and from Rossi's local maximum modulus principle follows that all polynomials attain their maximum modulus on $\overline{U} \setminus U$ (see Rossi [1] , (6.1)). A is therefore a maximum modulus algebra.

(4.1o) Goldmann [1] (5.5) has obtained an example of a doubly generated Montel function algebra, which does not have analytic structure at every point of its spectrum.
The crucial point in this example is that σA is not locally compact. By now we do not know any example of a finitely generated Montel function algebra with locally compact spectrum which is not a Stein algebra.

5. The range of the elements of a reflexive function algebra

In this section we consider - for reflexive function algebras - the range of non constant functions. First we prove a density property. By λ we denote the Lebesgue measure in the plane. We put $D(x,\delta) := \{z \in \mathbb{C} : |z-x| < \delta\}$.

(5.1) Theorem: *Let* $(A,\sigma A)$ *be a reflexive function algebra with connected spectrum* σA. *If* $f \in A$ *is not constant, then*

$$(*) \qquad \lim_{\delta\to 0} \frac{\lambda(f(\sigma A) \cap D(f(y),\delta))}{\lambda(D(f(y),\delta))} = 1$$

for all $y \in \sigma A$. *In particular,* $f(\sigma A) \cap D(f(y),\delta)$ *has positive Lebesgue measure, for all* $y \in \sigma A$ *and for all* $\delta > 0$.

Proof: Let $\tilde{K}_n \subset \tilde{K}_{n+1} \subset \dots$ be an admissible A-convex exhaustion of σA. Then by the operational calculus - which is also valid for function algebras (see Kramm [1] (7.1)) - $\varprojlim R(f(\tilde{K}_n))$ can be considered as a closed subalgebra of A, for every $f \in A$. Let B denote the closure in $\varprojlim R(f(\tilde{K}_n))$ of the set of all rational functions with poles outside of $f(\sigma(A))$. Then B is a reflexive function algebra. By point evaluation (resp. $\varphi \longmapsto \varphi(z)$ where z is the identity map in $\mathbb{C}$) we can identify σB with $f(\sigma(A))$.

We denote by L_n the B-convex hull of $f(\tilde{K}_n)$. Since the Gelfand topology on $\sigma B = f(\sigma(A))$ is finer than the induced euclidean topology, every L_n is a compact set in $\mathbb{C}$. Let V be a bounded component of $\mathbb{C} \setminus L_n$, then B- convexity implies that $V \cap (\mathbb{C} \setminus f(\sigma A))$ is not empty. Hence $B_{L_n} = R(L_n)$ and consequently $B = \varprojlim R(L_n)$.

If there is a point $y \in \sigma A$ such that (*) is not valid, then

$$\lim_{\delta\to 0} \frac{\lambda(L_n \cap D(f(y),\delta))}{\lambda(D(f(y),\delta))} < 1$$

for all $n \in \mathbb{N}$. From this follows that f(y) is a peak point for every algebra $R(L_n)$ (see Stout [1] (26.12)). Like in the proof of (4.4) we are now able to show that $\chi_{\{f(y)\}} \in B$. Because σA is connected, this implies that f is constant which contradicts our assumption.

We call a set $X \subset \mathbb{C}$ polynomially convex if for every compact subset $K \subset X$ the polynomially convex hull $\hat{K}$ is contained in X. The polynomially convex hull $\hat{X}$ of an arbitrary set $X \subset \mathbb{C}$ is the intersection of all polynomially convex sets Y containing X.

(5.2) Theorem: *Let* (A,X) *be a reflexive function algebra on a connected space* X. *Then for every non constant function* $f \in A$ $\widehat{f(X)}$ *is a (polynomially convex) domain in* $\mathbb{C}$.

Proof: Let $\tilde{K}_n \subset \tilde{K}_{n+1} \subset \ldots$ be an admissible exhaustion of σA. Then for every $f \in A$ $\varprojlim P(f(\tilde{K}_n)) = \varprojlim P(\widehat{f(\tilde{K}_n)})$ is a singly generated reflexive function algebra. Lemma (4.4) and Theorem (4.1) prove the assertion.

6. A weak identity theorem for maximum modulus algebras

(6.1) Definition: *A function algebra* (A,X) *satisfies the weak identity theorem in a point* $x \in X$, *if there exists a neighbourhood* U *of* x, *such that every function* $f \in A$, *which vanishes on some neighbourhood* V *of* x, *also vanishes on* U.

In his survey article [3] Kramm proves a weak identity theorem for strongly nuclear function algebras. We shall prove a similar result in the case of maximum modulus algebras. In our version a neighbourhood U in which every function must vanish is given explicitly.

We shall use the following result of Glicksberg (Glicksberg's lemma; see Gamelin [1] p. 39): Let (A,K) be a (Banach) function algebra on a compact set K, and let U be a non-empty open subset of K. Every function $f \in A$ which vanishes on U also vanishes on $\sigma A \setminus (\widehat{K \setminus U})$, where $\widehat{K \setminus U}$ denotes the A-convex hull of $K \setminus U$ in σA.

(6.2) Theorem: *Let* (A,X) *be a maximum modulus algebra. If there exists a compact neighbourhood* K *of* x *and a function* $f \in A$ *with* $\{f^{-1}(0)\} \cap K = \{x\}$ *then A satisfies in* x *the weak identity theorem.*

Proof: By assumption we have $\delta := \min_{\rho\in\partial K}|f(\rho)| > 0$. The subset $\{\psi \in K : |f(\psi)| \leq \delta/4\}$ is compact and contained in the interior of K. We define $V := \{\psi \in K : |f(\psi)| < \delta/4\}$. We choose an arbitrary point $\psi_0 \in V$ ($\psi_0 \neq x$) and set $\alpha := f(\psi_0)$. Then $0 < |\alpha| < \delta/4$.

For the function $g := f - \alpha$ we have

1) $\{g^{-1}(-\alpha)\} \cap K = \{x\}$
2) $\{\psi \in K : |g(\psi)| \leq |\alpha|\} =: K'$ is a compact subset of the interior of K
3) $g(\psi_0) = 0$.

Since A is a maximum modulus algebra the Shilov boundary $\gamma A_{K'}$ is contained in the topological boundary and therefore $g(\gamma A_{K'}) \subset g(\partial K') \subset \{z \in \mathbb{C} : |z| = |\alpha|\}$.

Let $W \subset \sigma A$ be an arbitrary neighbourhood of x, then $g(\gamma A_{K'} \setminus W)$ is polynomially convex, since $-\alpha \notin g(\gamma A_{K'} \setminus W)$. Thus we find a polynomial with

$$|p(0)| > \|p\|_{g(\gamma A_{K'} \setminus W)}.$$

For $p \circ g \in A$ this means

$$|p \circ g\,(\psi_0)| > \|p \circ g\|_{\gamma A_{K'} \setminus W}\ .$$

Hence ψ_0 is not in the $A_{K'}$-convex hull of $\gamma A_{K'} \setminus W$. Because of Glicksberg's lemma every function $h \in A$ which vanishes on W must also vanish in ψ_0. Since ψ_0 was an arbitrary point of V we are done.

Remark: The proof shows, that for compact K, $x \in K$ and $f \in A$ with $f(x) = 0$ and $\{f^{-1}(0)\} \cap \partial K = \emptyset$ every function $h \in A$, which vanishes on some neighbourhood W of $\{f^{-1}(0)\} \cap K$, also vanishes on V.

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THE HAHN-BANACH EXTENSION THEOREM FOR SOME SPACES OF n-HOMOGENEOUS POLYNOMIALS

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INTRODUCTION

S. Dineen asks in [6] the following question: "If E is a subspace of a locally convex space G, when can every holomorphic function on E be extended to a holomorphic function on G ?".

There are essentially two distinct cases of this problem: (1) E is a dense subspace of G; (2) E is a closed subspace of G.

Case (1) is the holomorphic analogue of finding the completion of a locally convex space and was discussed by Dineen, Hirschowitz, Noverraz, and others. Case (2) concerns an attempt to find a holomorphic Hahn-Banach theorem. We shall deal with this latter case. P. Boland proved in [4] that, if F is a closed subspace of a dual G of a nuclear Fréchet space, then every holomorphic function on F has an extension to a holomorphic function on G. This was the first general positive answer to the question. Afterwards Colombeau and Mujica gave in [5] another proof for this result. Using ideas of Theorem 4.5 of [12]*) and Theorem 4.1 of [5] it is possible to find an example of a dual G of a Fréchet Schwartz space which is not nuclear (but is Hilbertian) where Boland's assertion is still true. This shows that nuclearity is not necessary for extending all holomorphic functions in the case of (DF)-spaces. For metrizable spaces Vogt and Meise give in [11] an example of a nuclear Fréchet space G where the holomorphic Hahn-Banach theorem is not valid.

The case of Banach spaces was studied first by Aron and Berner [2] and Aron [1]. It is known that the holomorphic Hahn-Banach theorem is not valid in the general case of a Banach space, even in the case $G = E''$. For instance, if $E = c_0$ and $G = E'' = \ell_\infty$, there exist holomorphic mappings $f: c_0 \to \mathbb{C}$ which can not be extended to ℓ_∞ as a holomorphic mapping (see [7], 4.42). Aron and Berner proved in [2] that a holomorphic function $f: c_o \to \mathbb{C}$ can be extended to a holomorphic function on ℓ_∞ if and only if f is bounded on every bounded subset of c_o. Therefore it is reasonable to look for suitable "rich" classes of holomorphic mappings on E. Aron and Berner considered in [2] the problem of extending an analytic mapping defined on an open subset U of a closed subspace E of a Banach space G to an analytic mapping defined on an open neighborhood of U in G. Their general approach is to extend to the whole space G the n-homogeneous polynomials $\hat{d}^n f(y)$ defined on E, and then use local Taylor representations to extend the analytic function f locally. It is necessary to show that the local extensions are "coherent in the overlaps". This can be done if there is a linear and continuous mapping extending polynomials on E to polynomials on G. Clearly it is desirable to find types of holomorphy θ_1 and θ_2 such that there is a unique extension operator $H_{\theta_1}(E) \to H_{\theta_2}(G)$.

This note was motivated by these facts. We will consider some special classes of n-homogeneous polynomials on E and will extend the elements of these classes to n-homogeneous polynomials on E''. Moreover, we will characterize the space of the extended polynomials on E'', and will show that the extension mapping is linear, continuous and, in some sense, unique.

The proofs of the results contained in §2 are due to K. Floret and are much more simple than my original ones. They also improve my original results in the sense that they work also in the case: E metrizable and in the case: E'_β distinguished and metrizable. I gratefully acknowledge his communicating these results to me and his allowing me to include them here. Special thanks are due also to R. Aron for stimulating and helpful discussions concerning this work.

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NOTATION

Let E be a locally convex space. For all $n \in \mathbb{N}$, $\mathcal{L}(^nE)$ is the space of all continuous n-linear mappings from E^n into $\mathbb{K}$. The space $\mathcal{P}(^nE)$ of all continuous n-homogeneous polynomials on E is defined by

$$\mathcal{P}({}^nE) := \{\hat{A}: x \in E \mapsto A(x,\dots,x) \in \mathbb{K} \mid A \in \mathcal{L}({}^nE)\}.$$

We consider in $\mathcal{L}({}^nE)$ the topology β of the uniform convergence on the bounded subsets of E^n; and in $\mathcal{P}({}^nE)$ the topology β of the uniform convergence on the bounded subsets of E. The polarization formula establishes a topological isomorphism between $\mathcal{P}({}^nE)_\beta$ and the space $\mathcal{L}_s({}^nE)_\beta$ of the symmetric continuous n-linear mappings from E^n into $\mathbb{K}$.

It is well known that if E is a Banach space, $\mathcal{P}({}^nE)_\beta$ is a Banach space normed by $\hat{A} \mapsto \sup\{|\hat{A}(x)| : \|x\| \le 1\}$. $\mathcal{P}({}^nE)_\beta$ is also complete if E is metrizable or if E is a bornological (DF)-space (since in these cases n-linear mappings which are bounded on bounded sets are continuous; see Lemma 6 for the (DF)-case).

For other notations and basic results we refer to the book of Dineen [7].

§1: Let E be a locally convex space. The space of continuous n-homogeneous polynomials of finite type, denoted by $\mathcal{P}_f({}^nE)$, is the subspace of $\mathcal{P}({}^nE)$ spanned by $\{\varphi^n: x \in E \mapsto (\varphi(x))^n \mid \varphi \in E'\}$. Note that all the elements of $\mathcal{P}_f({}^nE)$ are weak continuous. The closure of $\mathcal{P}_f({}^nE)$ in $\mathcal{P}({}^nE)_\beta$ is denoted by $\mathcal{P}_c({}^nE)$. For details we refer to Gupta [9].

We define now:

$$\mathcal{P}_{f*}({}^nE'') := \operatorname{span}\{\varphi^n: x \in E'' \mapsto (\varphi(x))^n \mid \varphi \in E'\}$$

and

$$\mathcal{P}_{c*}({}^nE'') := \overline{\mathcal{P}_{f*}({}^nE'')} = \text{closure of } \mathcal{P}_{f*}({}^nE'') \text{ in } \mathcal{P}({}^nE'')_\beta\,;$$

clearly β points now at the uniform convergence on all bounded subsets of $(E'_\beta)'_\beta$.

Note also that all the elements of $\mathcal{P}_{f*}({}^nE'')$ are weak*-continuous.

LEMMA 1. If E is a Banach space and $n \in \mathbb{N}$, then there exists for every $P \in \mathcal{P}_c({}^nE)$ a unique extension $\tilde{P} \in \mathcal{P}_{c*}({}^nE'')$. The operator defined by $T_nP := \tilde{P}$ has the following properties:

(1) $\|T_n\| = 1$ for all $n \in \mathbb{N}$;

(2) T_n is an isomorphism from $\mathcal{P}_c({}^nE)$ onto $\mathcal{P}_{c*}({}^nE'')$;

(3) for every k, $m \in \mathbb{N}$, $k \leq m$ and $P \in \mathcal{P}_c({}^mE)$,

$$\hat{d}^k(T_mP)(y) = T_k\hat{d}^kP(y) \quad \text{for all } y \in E.$$

PROOF. For each $\varphi \in E'$ define $\tilde{\varphi}(x) := x(\varphi)$ for every $x \in E''$. Clearly $\tilde{\varphi} \in (E'', \sigma(E'',E'))'$. Every $P = \sum_{i=1}^{p} \varphi_i^n$ is $\sigma(E,E')$-continuous and since E is $\sigma(E'',E')$-dense in E'' it has at most one $\sigma(E'',E')$-continuous extension $\tilde{P}$ to E'', obviously $\tilde{P} := \sum_{i=1}^{p} \tilde{\varphi}_i^p$. Moreover, since the unit ball of E is $\sigma(E'',E')$-dense in the unit ball of E'', the extension mapping $P \mapsto \tilde{P}$ is a linear surjective isometry. Since $\mathcal{P}_c({}^nE)$ and $\mathcal{P}_{c*}({}^nE'')$ are Banach spaces, it can be extended to an surjective isometry

$$T_n\colon \overline{\mathcal{P}_f({}^nE)} = \mathcal{P}_c({}^nE) \to \overline{\mathcal{P}_{f*}({}^nE'')} = \mathcal{P}_{c*}({}^nE'').$$

Now for all $\varphi \in E'$, $y \in E$ and $k \in \mathbb{N}$, $0 \leq k \leq m$:

$$\hat{d}^kT_m(\varphi^m)(y) = k!\binom{m}{k}(\tilde{\varphi}(y))^{m-k}\,\tilde{\varphi}^k =$$

$$= T_k[k!\binom{m}{k}(\varphi(y))^{m-k}\,\varphi^k] = T_k\hat{d}^k(\varphi^m)(y).$$

Since the operators T_k, T_m and $P \to \hat{d}^kP(y)$ are all continuous and linear and since the span of polynomials of the form φ^n is dense in $\mathcal{P}_c({}^nE)$, the statement (3) is proved as well. ■

REMARK 2. If F is a Banach space and we define $\mathcal{P}_c({}^nE,F) := \overline{\mathcal{P}_f({}^nE) \otimes F}$, closure in $\mathcal{P}({}^nE;F)_\beta$; $\mathcal{P}_{c*}({}^nE'',F) := \overline{\mathcal{P}_{f*}({}^nE'') \otimes F}$, closure in $\mathcal{P}({}^nE'',F)_\beta$, Lemma 1 can be proved by taking $\mathcal{P}_c({}^nE,F)$ and $\mathcal{P}_{c*}({}^nE'',F)$ instead of $\mathcal{P}_c({}^nE)$ and $\mathcal{P}_{c*}({}^nE'')$ respectively.

PROPOSITION 3. Let E be a strict inductive limit of Banach spaces E_k. Then for every $P \in \mathcal{P}_c({}^nE)$ there is a unique extension $\tilde{P} \in \mathcal{P}_{c*}({}^nE'')$. The operator

$$T_n\colon \mathcal{P}_c({}^nE) \to \mathcal{P}_{c*}({}^nE''), \quad P \mapsto \tilde{P}$$

is a surjective isomorphism.

PROOF. First observe that whenever F is a subspace of G the definitions (approximating the polynomials by "elementary" polynomials $\sum_{i=1}^{p} \varphi_i^n$) imply that the following diagram (all mappings being the respective restriction mappings)

$$(*) \qquad \begin{array}{ccc} \mathcal{P}_{c*}({}^nG'') & \to & \mathcal{P}_c({}^nG) \\ \downarrow & & \downarrow \\ \mathcal{P}_{c*}({}^nF'') & \to & \mathcal{P}_c({}^nF) \end{array}$$

is well-defined and commutative.

Now take $E = \operatorname{ind}_{k\to} E_k$ (strict inductive limit of Banach spaces E_k); by [8] p. 86 the limit of the strict inductive sequence (E_k'') is E_β''.

For $P \in \mathcal{P}_c({}^nE)$ define $P_k := P|_{E_k} \in \mathcal{P}_c({}^nE_k)$ (by (*)) and take the unique extension $\tilde{P}_k \in \mathcal{P}_{c*}({}^nE_k'')$ of P_k. Then, again by (*) and the uniqueness of the extension

$$\tilde{P}_{k+1}\Big|_{E_k''} = \tilde{P}_k ,$$

and the following definition of an extension $\tilde{P}$ of P is possible:

$$\tilde{P}(x) := \tilde{P}_k(x) \quad \text{if} \quad x \in E_k'' .$$

Since every bounded set $B \subset E_\beta''$ is bounded in some Banach-space E_k'', the Hahn-Banach theorem implies that on B the polynomial $\tilde{P}$ (which is actually $\tilde{P}_k$) can be uniformly approximated by $\sum_{i=1}^{p} \varphi_i^n$ with $\varphi_i \in E'$. Since the extension operators on all steps E_k are surjective linear isometries it follows that

$$T_n : \mathcal{P}_c({}^nE) \to \mathcal{P}_{c*}({}^nE'')$$
$$P \mapsto \tilde{P}$$

is a surjective isomorphism. ∎

§2: For a locally convex space E we define

$$\mathcal{P}_{wu}({}^nE) := \{P \in \mathcal{P}({}^nE) : P|_B \text{ is } \sigma(E,E')\text{-uniformly continuous for every } B \subset E \text{ bounded}\}$$

$$\mathcal{P}_{w*u}({}^nE'') := \{P \in \mathcal{P}({}^nE'') : P|_{B^{oo}} \text{ is } \sigma(E'',E')\text{-uniformly continuous for every } B \subset E \text{ bounded}\}$$

We note that $\mathcal{P}_{wu}({}^nE) \neq \mathcal{P}({}^nE)$, even in the case of Banach spaces. To show this, take the following example from [3]: if we define $P(x) := \sum_{n=1}^{\infty} x_n^2$ for all $x = (x_n) \in \ell_2$, it is clear that $P \in \mathcal{P}(^2\ell_2)\backslash\mathcal{P}_{wu}(^2\ell_2)$ since the canonical basis vectors (e_n) in ℓ_2 tend to zero weakly but $P(e_n) = 1$ for all n.

If E is a Banach space, Theorem 2.9 of [3] shows that

$$\mathcal{P}_{wu}(^nE) = \{P \in \mathcal{P}(^nE) : P|_B \text{ is } \sigma(E,E')\text{-continuous for every } B \subset E \text{ bounded}\}.$$

We intend to construct extension mappings

$$\hat{T}_n : \mathcal{P}_{wu}(^nE) \to \mathcal{P}_{w^*u}(^nE'').$$

To do this, we define

$$\mathcal{L}_a(^nE) := \{A : E^n \to \mathbb{K} \mid n\text{-linear}\}$$

$$\mathcal{L}_{a,wu}(^nE) := \{A \in \mathcal{L}_a(^nE) \mid \text{for every } B \subset E \text{ bounded, } A|_{B^n} \text{ is uniformly continuous on } (B,\sigma(E,E'))^n\}.$$

Then the following result holds:

PROPOSITION 4. Let E be a locally convex space. Then for every $A \in \mathcal{L}_{a,wu}(^nE)$ there is a unique $\tilde{A} \in \mathcal{L}_a(^nE'')$ such that for every B bounded in E the restriction of $\tilde{A}$ to $(B^{oo},\sigma(E'',E'))^n$ is uniformly continuous. Moreover

$$\sup_{x\in B^n} |A(x)| = \sup_{x\in (B^{oo})^n} |\tilde{A}(x)|$$

for all bounded subsets $B \subset E$ and the mapping

$$T_n : \mathcal{L}_{a,wu}(^nE) \to \mathcal{L}_a(^nE''), \quad A \mapsto \tilde{A}$$

is linear and injective. If $A \in \mathcal{L}_{a,wu}(^nE)$ is symmetric $\tilde{A}$ is symmetric as well.

PROOF. Take $A \in \mathcal{L}_{a,wu}(^nE)$. Since uniformly continuous functions on dense subsets of uniform spaces have a unique uniformly continuous extension to the whole space and for a given absolutely convex bounded set $B \subset E$ the set B^n is dense in $(B^{oo},\sigma(E'',E'))^n$ there is a unique uniformly continuous function $\tilde{A}^B$ on $(B^{oo})^n$ which extends $A|_{B^n}$. By the uniqueness of extensions

$$\tilde{A}^C \Big|_{(B^{oo})^n} = \tilde{A}^B$$

whenever $B \subset C$ and this shows that

$$\tilde{A}(x) := \tilde{A}^B(x) \quad \text{if} \quad x \in B^{oo}$$

defines an n-linear map $\tilde{A}$ on $(E'')^n = (\bigcup \{B^{oo} \mid B \subset E$ bounded and absolutely convex$\})^n$.

The other statements are obvious by the density and the uniqueness of the extension. ■

Obviously, if E is normed then $A \in \mathcal{L}_{a,wu}(^nE)$ is norm-continuous if and only if $\tilde{A}$ is norm-continuous and

$$\|A\| = \|\tilde{A}\|.$$

Since there is a 1-1 correspondence $\hat{}$ between symmetric n-linear mappings and n-homogeneous polynomials, the definition

$$\hat{T}_n\hat{A} := (T_nA)\hat{}$$

gives the

COROLLARY 5. If E is a normed space then there is an isomorphism (onto)

$$\hat{T}_n: \mathcal{P}_{wu}(^nE) \to \mathcal{P}_{w*u}(^nE'')$$

such that

(1) $\quad \hat{T}_nP\big|_E = P$ for all $P \in \mathcal{P}_{wu}(^nE)$;

(2) $\quad \|T_n\| \leq \frac{n^n}{n!}$.

For the proof of (2) take a symmetric continuous $A \in \mathcal{L}_{a,wu}(^nE)$; then

$$\|\hat{T}_n\hat{A}\| = \|(T_nA)\hat{}\| \leq \|T_nA\| = \|A\| \leq \frac{n^n}{n!}\|\hat{A}\|. \qquad ■$$

This extension result can also be generalized to other cases. We shall use the following

LEMMA 6. Let $E_1,\ldots,E_n$ be bornological (DF)-spaces. Then every n-linear mapping on $E_1 \times\ldots\times E_n$ which is bounded on bounded sets is continuous.

PROOF. We prove this lemma by induction: For $n = 1$ it is obvious.

So assume it is true for $n-1$, take $A: E_1 \times\ldots\times E_n \to \mathbb{K}$ as stated and fix $(x_3,\ldots,x_n) \in E_3 \times\ldots\times E_n$. Then $A(\cdot,\cdot\cdot,x_3,\ldots,x_n): E_1 \times E_2 \to \mathbb{K}$ is hypocontinuous since for $B_1 \subset E_1$ bounded, the set $\{x_2 \in E_2 : |A(b,x_2,x_3,\ldots,x_n)| \leq 1$ for all $b \in B_1\}$ is absolutely convex and absorbs bounded sets and is therefore a neighborhood in E_2 (same reasoning for $B_2 \subset E_2$). Since E_1 and E_2 are (DF)-spaces $A(\cdot,\cdot\cdot,x_3,\ldots,x_n)$ is continuous and $A_1: (E_1 \otimes_\pi E_2) \times E_3 \times \ldots \times E_n \to \mathbb{K}$ is well defined. Note that $E_1 \otimes_\pi E_2$ is a bornological (DF)-space (see [10] 15.5.4 (b)).

To apply the induction-hypothesis we have to show that A_1 is bounded on bounded sets: For $B \subset E_1 \otimes_\pi E_2$ bounded there are $B_1 \subset E_1$ and $B_2 \subset E_2$ such that $B \subset \overline{\Gamma(B_1 \otimes B_2)}$ (see e.g. the proof of [10] 15.6.2). Whence if $B_3 \subset E_3, \ldots, B_n \subset E_n$ are bounded

$$A_1(B,B_3,\ldots,B_n) \subset A_1(\overline{\Gamma(B_1\otimes B_2)},B_3,\ldots,B_n) \subset \overline{A_1(\Gamma(B_1\otimes B_2),B_3,\ldots,B_n)}$$

(since $A_1(\cdot,x_3,\ldots,x_n)$ is always continuous) and $\overline{A_1(\Gamma(B_1\otimes B_2),B_3,\ldots,B_n)} \subset \overline{\Gamma A(B_1,B_2,B_3,\ldots,B_n)}$ which is bounded. ■

PROPOSITION 7. Let E be a locally convex space such that

(a) E'_β is distinguished metrizable, or

(b) E is a strict inductive limit of Banach spaces, or

(c) E is metrizable.

Then, for every $n \in \mathbb{N}$ there is a unique isomorphism (onto)

$$\hat{T}_n : \mathcal{P}_{wu}(^nE) \to \mathcal{P}_{w^*u}(^nE'')$$

such that

(1) $\hat{T}_n P\big|_E = P$ for all $P \in \mathcal{P}_{wu}(^nE)$

(2) $\sup_{x\in B^{oo}} |(\hat{T}_n P)(x)| \leq \frac{n^n}{n!} \sup_{x\in B} |P(x)|$ for all bounded and absolutely convex subsets $B \subset E$.

PROOF. As before we shall define $\hat{T}_n\hat{A} := (T_nA)^\wedge$ for every $A \in \mathcal{L}_{wu}(E)$. So all we have to show is that $\hat{A}$ is continuous, whenever A is continuous.

(a) If E'_β is distinguished and metrizable, the second dual E''_β is a bornological (DF)-space such that all bounded sets are equicontinuous whence contained in some B^{oo} with $B \subset E$ bounded.

This implies that for every continuous $A \in \mathcal{L}_{a,wu}(^nE)$ its extension $\tilde{A}$ given by Proposition 4 is bounded on bounded sets and therefore continuous by Lemma 6.

(b) If E is the strict inductive limit of Banach spaces E_k then (by [8], p. 86) E''_β is also a strict inductive limit of Banach spaces, in particular barrelled, whence E'_β is distinguished - and clearly metrizable. So (b) is a special case of (a).

(c) If E is metrizable and $A \in \mathcal{L}_{a,wu}(^nE)$ is continuous, then the extension $\tilde{A}$, given by Proposition 4, is bounded on sets of the form B^{oo} where $B \subset E$ is bounded. Since zero-sequences in the strong dual of a (DF)-space are equicontinuous, this implies that $\tilde{A}$ is bounded on zero-sequences in $(E''_\beta)^n$. Since this latter space is metrizable, this easily implies that $\tilde{A}$ is continuous in zero and so continuous. ∎

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*) NOTE.

Section 4 of [12] has been separated from the rest of this preprint. Hence Theorem 4.5 of [12] will appear as Theorem 6 of the following article:

Meise, R. and Vogt, D. - Extension of entire functions on nuclear locally convex spaces. Preprint (1983).

Functional Analysis: Surveys and Recent Results III
K.D. Bierstedt and B. Fuchssteiner (eds.)

A GENERALIZATION OF A THEOREM OF KELDYŠ

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Let U be an open subset of $\mathbb{C}$, $\mu \in U$, E and F Banach spaces and T be a holomorphic operator function on U with values in L(E,F). Suppose that the nullspace $N(T(\mu))$ is finite-dimensional, that T^{-1} exists on $U \setminus \{\mu\}$ and that the singularity of T^{-1} at μ is a pole of positive order. Under these assumptions the operators in the singular part of T^{-1} at μ are finite-dimensional and representable by canonical systems of eigenvectors and associated vectors of T and T', where T' is the adjoint operator function of T. The eigenvectors and the associated vectors belonging to these canonical systems fulfill certain bi-orthogonal relationships.

If U is a domain, T is a holomorphic Fredholm operator function and $T(\lambda_0)$ is bijective for some $\lambda_0 \in U$, then $\sigma(T)$ is a discrete subset of U and T^{-1} is a meromorphic function with poles at all points of $\sigma(T)$. Therefore, under these stronger conditions on U and T, the above mentioned assumptions on T^{-1} are fulfilled a priori if, without any loss of generality, we substitute U by $(U \setminus \sigma(T)) \cup \{\mu\}$. T is Fredholm-valued if E = F and $T(\lambda) = id_E + K(\lambda)$ where $K(\lambda)$ is a compact operator in E for $\lambda \in U$. For operator functions of this type, acting in a Hilbert space, the above stated representation theorem is due to Keldyš, cf. [8], [9].

In [3] Gohberg and Sigal established a factorization theorem for meromorphic operator functions. As an application they proved Keldyš's representation theorem (apart from the biorthogonality relationships for the eigenvectors and associated vectors) for holomorphic Fredholm operator functions. Instead of eigenvectors and associated vectors, Gohberg and Sigal used the notion of root functions, which considerably simplified the formulas and made the theorem as well as its proof more transparent.

In [12] the authors gave a direct proof of Keldyš's representation theorem, which they stated in the simpler notation of root functions as well as in the original form with respect to eigenvectors and associated vectors.

In the present paper the authors give a generalization of Keldyš's theorem for holomorphic operator functions acting between Frêchet spaces E and F. The proof of this generalization is closely related to the authors' Banach space proof in [12] which, fortunately, did not make use of Gohberg and Sigal's factorization theorem; a theorem of this type is not (yet) available for operator functions acting in Frêchet spaces.

Representation theorems concerning the local analytic structure of the resolvent T^{-1} are fundamental for expansions in series of eigenvectors and associated vectors; they have important applications to boundary eigenvalue problems, as they yield representations of the singular part of the Green functions in terms of eigenvectors and associated vectors. The theorem proved in this paper allows applications to eigenvalue problems for differential equations in the complex domain of which we briefly treat an example in the final section.

1. Holomorphic operator functions in F-spaces.

Let E and F be Frêchet spaces and L(E,F) be the vector space of all continuous linear operators from E into F. If $A \in L(E,F)$, N(A) denotes its nullspace and R(A) its range. A is called a Fredholm operator if both its nullity nul(A) := dim N(A) and its deficiency def(A) := codim R(A) are finite. $\phi(E,F)$ denotes the set of all Fredholm operators from E into F. We set

$$(y \otimes v)(w) := \langle w,v \rangle y \qquad (w \in F)$$

for $y \in E$ and $v \in F'$, and state that the tensor product $y \otimes v$ belongs to L(F,E).

Let U be an open subset of $\mathbb{C}$ and X: U → E. X is called holomorphic if it is differentiable in U, which is equivalent to the fact that for each $\mu \in U$ there are $x_l \in E (l \in \mathbb{N})$, such that

$$(1.1) \qquad X(\lambda) = \sum_{l=0}^{\infty} (\lambda-\mu)^l x_l$$

where the series converges in E for all λ in some neighborhood of μ. X is holomorphic iff it is weakly holomorphic which means that the function $\langle X(.),x'\rangle : U \to \mathbb{C}$ is holomorphic for all $x' \in E'$. (Cf. Kaballo [7], chap. I and Uss [14], p. 149.) We denote by H(U,E) the set of all holomorphic functions from U into E.

If B is a bounded subset of E and q is a continuous seminorm on F then

$$p_{B,q}(A) := \sup_{x \in B} q(Ax) \qquad (A \in L(E,F))$$

defines a seminorm on L(E,F). $L_b(E,F)$ denotes the space L(E,F) equipped with the Hausdorff locally convex topology generated by the set of all seminorms $p_{B,q}$.

An operator function $T: U \to L(E,F)$ is called holomorphic if it is differentiable as a function from U into $L_b(E,F)$. This property is equivalent to the fact that for each $\mu \in U$ there are $T_l(\mu) \in L(E,F)$ $(l \in \mathbb{N})$ such that

$$(1.2) \qquad T(\lambda) = \sum_{l=0}^{\infty} (\lambda-\mu)^l T_l(\mu)$$

where the series converges in $L_b(E,F)$ for all λ in some neighborhood of μ. T is holomorphic iff T is weakly holomorphic which means that the function $\langle T(.)x,y'\rangle : U \to \mathbb{C}$ is holomorphic for all $x \in E$ and $y' \in F'$. (Cf. Kaballo [7] and Uss [14].) If $T \in H(U,L(E,F))$, i.e. T is holomorphic, then T is indefinitely differentiable with the l-th derivative $T^{(l)}(\mu) = l!T_l(\mu)$.

Let $T \in H(U,L(E,F))$ and $S \in H(U,L(G,E))$ where also G is some Frêchet space. It is easy to prove that $TS \in H(U,L(G,F))$ and

$$(1.3) \qquad (TS)_l(\mu) = \sum_{k=0}^{l} T_k(\mu) S_{l-k}(\mu) \qquad (l \in \mathbb{N}).$$

It follows that TX is holomorphic if $T \in H(U,L(E,F))$ and $X \in H(U,E)$. X is called a *root function* of T at μ if $X(\mu) \neq 0$ and $(TX)(\mu) = 0$. If X is a root function of T at μ, then $\nu(X)$ denotes the order of the zero of TX at μ and is called the multiplicity of X (with respect to T at μ). A system $\{X_1, X_2, \ldots, X_r\}$ of root functions of T at μ is called a *canonical system of root functions* (CSRF) if

$$(1.4) \qquad \{X_1(\mu), X_2(\mu), \ldots, X_r(\mu)\} \text{ is a basis of } N(T(\mu))$$

and for all $j\in\{1,2,\dots,r\}$

(1.5) $\nu(X_j)$ is the maximum of all $\nu(X)$ where X varies in the set of root functions of T at μ such that

$$X(\mu)\in N(T(\mu)) \setminus \operatorname{span}\{X_1(\mu),\dots,X_{j-1}(\mu)\}.$$

It is easy to show that there is a CSRF of T at μ if i) the nullity nul $T(\mu)$ is finite and ii) the inverse operator function T^{-1} belongs to $H(U\setminus\{\mu\},L(F,E))$ and has a pole of order $s \geq 1$ at μ (which means that $(.-\mu)^sT^{-1}$ is holomorphic in U and different from zero at μ).

An ordered set $\{x_0,x_1,\dots,x_h\} \subset E$ is called a *chain of an eigenvector and associated vectors* (CEAV) of T at μ if

$$X := \sum_{l=0}^{h} (.-\mu)^l x_l$$

is a root function of T at μ with $\nu(X) \geq h+1$. We denote by $\overline{\nu}(x_0)$ the maximum of all $\nu(X)$ where X is a root function of T at μ with $X(\mu)=x_0$.

Conversely, if X is a root function of T at μ, $\nu(X) \geq h+1$ and X is expanded in a series of the form (1.1) then the Taylor coefficients $x_0,\dots,x_h$ form a CEAV.

A system $\{x_l^{(j)}: 1 \leq j \leq r,\ 0 \leq l \leq m_j-1\}$ is called a *canonical system of eigenvectors and associated vectors* (CSEAV) of T at μ if

(1.6) $\{x_0^{(j)}: 1 \leq j \leq r\}$ is a basis of $N(T(\mu))$,

(1.7) $\{x_0^{(j)},x_1^{(j)},\dots,x_{m_j-1}^{(j)}\}$ is a CEAV of T at μ $(j=1,2,\dots,r)$,

(1.8) $\overline{\nu}(x_0^{(j)}) = \sup\{\overline{\nu}(x_0): x_0\in N(T(\mu))\setminus\operatorname{span}\{x_0^{(k)}:k<j\}\}$ $(1\leq k\leq r)$.

If we define

$$X_j := \sum_{l=0}^{m_j-1} (.-\mu)^l x_l^{(j)} \qquad (j=1,2,\dots,r)$$

then a system $\{x_l^{(j)}: 1 \leq j \leq r,\ 0 \leq l \leq m_j - 1\}$ is a CSEAV of T at μ iff $\{X_j: 1 \leq j \leq r\}$ is a CSRF of T at μ with $\nu(X_j) = m_j$ $(j=1,2,\dots,r)$.

$\rho(T) := \{\lambda\in U: T(\lambda)$ bijective$\}$ is called the resolvent set of $T\in H(U,L(E,F))$, $\sigma(T) := U\setminus\rho(T)$ its spectrum and $\sigma_p(T) := \{\lambda\in U: \text{nul } T(\lambda)\neq 0\}$ its point spectrum. The inverse operator function T^{-1}, the resolvent of T, is defined pointwise by $T^{-1}(\lambda) := (T(\lambda))^{-1}$ $(\lambda\in\rho(T))$.

If E and F are Banach spaces then $\rho(T)$ is an open subset of U and $T^{-1}\in H(\rho(T),L(F,E))$. If, in addition, U is connected, i.e. a domain in $\mathbb{C}$, $T(\lambda)\in\phi(E,F)$ for all $\lambda\in U$ and $\rho(T) \neq \emptyset$ then $\sigma(T)$ is a discrete subset of U, $\sigma(T) = \sigma_p(T)$, and T^{-1} is meromorphic and has poles at all points of $\sigma(T)$ (cf. Gramsch [5] or Bart [2]).

Similar statements do not hold when E and F are Fréchet spaces. Pietsch [13], p. 355, gave a simple example which shows that $\rho(T)$ is not necessarily open. Even if $\rho(T)$ is an open set it can happen that T^{-1} is not holomorphic on $\rho(T)$, an assertion which is proved by the following

(1.9) <u>Example</u>. $E = \{y\in C^\infty(\mathbb{R}): \forall i\in\mathbb{N}\ \ y^{(i)}(0) = 0\}$ is a closed subspace of the Fréchet space $C^\infty(\mathbb{R})$ and therefore also a Fréchet space. Let $T(\lambda)y = y+\lambda y'$ for $y\in E$. Obviously $T\in H(\mathbb{C},L(E))$ and $\rho(T) = \mathbb{C}$. If $\lambda \neq 0$ and $g\in E$ then

$$(T^{-1}(\lambda)g)(x) = \int_0^x \frac{1}{\lambda}\exp\{\frac{1}{\lambda}(\eta-x)\}g(\eta)d\eta \qquad (x\in\mathbb{R}).$$

Let $g(x) = \exp(-\frac{1}{x^2})$ if $x\in\mathbb{R}\setminus\{0\}$ and $g(0) = 0$. The function g belongs to E and $C := \inf\{g'(x): x\in[1,2]\} > 0$. We set $B = \{g\}$ and $q(y) = |y(3)|$ for $y\in E$. We infer that

$$p_{B,q}(T^{-1}(\lambda)-T^{-1}(0)) = \left|\int_0^3 e^{\frac{1}{\lambda}(\eta-3)} g'(\eta)d\eta\right|$$

$$\geq Ce^{-\frac{1}{\lambda}} \to \infty \text{ if } \lambda < 0 \text{ and } \lambda \to 0$$

which means that the operator function T^{-1} is not continuous and thus not holomorphic at 0.

The above cited example of Pietsch also shows that the Banach space statements on the discreteness of $\sigma(T)$ and the meromorphy of T^{-1} do not hold in Fréchet spaces. Concerning some positive results of this kind we refer to Kaballo [7], ch. I, and Mennicken [11], sec. 3.3.

2. Main results.

Let E_1,E_2,F_1,F_2 be Fréchet spaces. Assume that (E_1,E_2) and (F_1,F_2) are dual pairs with regard to bilinear forms $<\ ,\ >$ on $E_1\times E_2$ or $F_1\times F_2$ respectively. The bilinear forms are supposed to be separately continuous so that each of the Fréchet spaces under study can be identified with a subspace of the dual of the second space with which it forms the dual pair. In this sense, for example, E_2 is a subspace of E_1'.

(2.1) Proposition. Let N be a finite-dimensional subspace of E_1 and $\{x_1,x_2,\dots,x_k\}$ be a basis of N. Suppose that $A\in L(F_1,E_1)$ and that $R(A)\subset N$ and $R(A'|_{E_2})\subset F_2$ where $A'|_{E_2}$ denotes the restriction of the adjoint operator A' to E_2.

Then there are $v_1,\dots,v_k \in F_2$ such that

$$A = \sum_{i=1}^{k} x_i \otimes v_i .$$

Proof. Since (E_1,E_2) is a dual pair we can choose $x_j'\in E_2$ such that $\langle x_i,x_j'\rangle = \delta_{ij}$ for $i,j = 1,2,\dots,k$. Define $v_j := A'x_j'$; by assumption $v_j\in F_2$. Because $R(A) \subset N$ we have

$$Ay = \sum_{i=1}^{k} \alpha_i(y)x_i \qquad (y\in F)$$

with $\alpha_i(y)\in\mathbb{C}$. We conclude that

$$\langle y,v_j\rangle = \langle y,A'x_j'\rangle = \langle Ay,x_j'\rangle = \alpha_j(y)$$

whence

$$Ay = \sum_{i=1}^{k} \langle y,v_i\rangle x_i = \sum_{i=1}^{k} (x_i\otimes v_i)(y).$$

(2.2) Proposition. Assume that $X_i\in H(U,E_1)$ and $Z_i\in H(U\setminus\{\mu\},F_2)$ for $i\in\{1,2,\dots,r\}$. Suppose that $X_1(\mu),\dots,X_r(\mu)$ are linearly independent and that the Z_i have (at most) a pole at μ.

We assert that $Z_i\in H(U,F_2)$ for all $i\in\{1,2,\dots,r\}$ if

$$\sum_{i=1}^{r} X_i\otimes Z_i\in H(U,L(F_1,E_1)).$$

Proof. Choose the smallest natural number s, such that all the functions $\hat{Z}_i := (\cdot-\mu)^s Z_i$ are holomorphic in U. Suppose that $s \geq 1$. Let $y\in F_1$. By assumption the function

$$X(\lambda) := \sum_{i=1}^{r} \langle y,Z_i(\lambda)\rangle\, X_i(\lambda) \qquad (\lambda\in U)$$

is holomorphic in U. Since

$$(\lambda-\mu)^s\, X(\lambda) = \sum_{i=1}^{r} \langle y,\hat{Z}_i(\lambda)\rangle\, X_i(\lambda)$$

and $s \geq 1$, we infer

$$0 = \sum_{i=1}^{r} \langle y, \hat{Z}_i(\mu)\rangle \; X_i(\mu)$$

and further, by the linear independence of the $X_i(\mu)$,

$$\langle y, \hat{Z}_i(\mu)\rangle = 0$$

for all $i \in \{1,2,\dots,r\}$. As y is arbitrary in F_1, all $\hat{Z}_i(\mu)$ vanish, which contradicts the minimality of s.

From now on we assume that $T \in H(U, L(E_1,F_1))$, $T^* \in H(U, L(F_2,E_2))$ and that these two operator functions are adjoint to each other in the following sense:

$$\langle T(\lambda)x, y'\rangle = \langle x, T^*(\lambda)y'\rangle$$

for all $x \in E_1, y' \in F_2$ and $\lambda \in U$.

(2.3) Lemma. *Let* $\mu \in \sigma(T)$. *Suppose that* $T^{-1} \in H(U\setminus\{\mu\}, L(F_1,E_1))$ *and* $T^{*-1} \in H(U\setminus\{\mu\}, L(E_2,F_2))$. *Assume that the singularity of* T^{-1} *at* μ *is a pole. Let* $\{X_1,\dots,X_r\}$ *be a* CSRF *of* T *at* μ. *We set* $m_j := \nu(X_j)$ $(j=1,\dots,r)$.

Then there are polynomials $V_j: \mathbb{C} \to F_2$ *of degree less than* m_j *and an operator function* $D \in H(U, L(F_1,E_1))$, *such that*

$$T^{-1}(\lambda) = \sum_{j=1}^{r} (\lambda-\mu)^{-m_j} \; X_j(\lambda) \otimes V_j(\lambda) + D(\lambda) \tag{2.4}$$

for all $\lambda \in U\setminus\{\mu\}$. *The* V_j *are uniquely determined by the system* $\{X_1,\dots,X_r\}$.

$\{V_1,\dots,V_r\}$ *is a* CSRF *of* T^* *at* μ, $\nu(V_j) = m_j$ *and the biorthogonal relationships*

$$\frac{1}{l!}\frac{d^l}{d\lambda^l} \langle \eta_{ih}, V_j\rangle(\mu) = \delta_{ij}\delta_{m_i-h,l} \tag{2.5}$$

$$(1 \leq h \leq m_i,\; 0 \leq l \leq m_j - 1,\; 1 \leq i,j \leq r)$$

hold where

$$\eta_{ih}(\lambda) := (\lambda-\mu)^{-h}(TX_i)(\lambda) \qquad (\lambda \in U). \tag{2.6}$$

Proof. At first we prove the following assertion:

(2.7) Let s be the pole order of T^{-1} at μ. For $\kappa = 0,1,\ldots,s$ there are polynomials $V_j^\kappa : \mathbb{C} \to F_2$ of degree less than m_j such that the pole order of

$$(2.8)\qquad T^{-1} - \sum_{j=1}^{r} (\cdot-\mu)^{-m_j}\, X_j \otimes V_j^\kappa$$

at μ does not exceed $s-\kappa$.

For $\kappa = 0$ (2.7) holds if we set $V_j^0 = 0$ $(j=1,\ldots,r)$. Assume that (2.7) is fulfilled for some $0 \le \kappa < s$. We set

$$(2.9)\qquad A(\lambda) := \sum_{l=-s+\kappa}^{\infty} (\lambda-\mu)^l A_l := T^{-1}(\lambda) - \sum_{j=1}^{r} (\lambda-\mu)^{-m_j} X_j(\lambda) \otimes V_j^\kappa(\lambda)$$

(where the expansion is valid in some neighborhood of μ). The functions $(\cdot-\mu)^{-m_j}\, TX_j$ are holomorphic at μ since $m_j = \nu(X_j)$. From (2.9) we conclude that TA is holomorphic at μ. Let $x \in R(A_{-s+\kappa}) \setminus \{0\}$. Choose some $y \in F_1$ such that $x = A_{-s+\kappa} y$ and define $X(\lambda) := (\lambda-\mu)^{s-\kappa} A(\lambda) y$ $(\lambda \in U)$. X is a root function of T at μ with $X(\mu) = x$ and $\nu(X) \ge s-\kappa$. This proves that $R(A_{-s+\kappa}) \subset L_{s-\kappa}$ where

$$L_{s-\kappa} := \{X(\mu) : X \text{ root function of } T \text{ at } \mu,\ \nu(X) \ge s-\kappa\} \cup \{0\}.$$

Since $\{X_1,\ldots,X_r\}$ is a CSRF of T at μ we conclude that

$$L_{s-\kappa} = \operatorname{span}\{X_j(\mu) : m_j \ge s-\kappa\}.$$

It is easy to show that $T^{-1}(\lambda)'|_{E_2}$, i.e. the restriction to E_2 of the adjoint operator of $T^{-1}(\lambda)$, is equal to $T^{*-1}(\lambda)$ for all $\lambda \in U \setminus \{\mu\}$. Therefore, and because $V_j^\kappa(\lambda) \in F_2$ for all $j \in \{1,2,\ldots,r\}$, the mapping $\lambda \to A(\lambda)'|_{E_2}$ defines a holomorphic operator function on $U \setminus \{\mu\}$ with values in $L(E_2,F_2)$. From this we infer that $R(A_l'|_{E_2}) \subset F_2$ for all $l \ge -s+\kappa$. Proposition (2.1), applied with respect to $N = L_{s-\kappa}$ and $A = A_{-s+\kappa}$, yields

$$(2.10)\qquad A_{-s+\kappa} = \sum_{j=1}^{r} X_j(\mu) \otimes v_j$$

with $v_j \in F_2$ and $v_j = 0$ if $m_j < s-\kappa$. We set

$$V_j^{\kappa+1}(\lambda) := V_j^\kappa(\lambda) + (\lambda-\mu)^{m_j-s+\kappa}\, v_j \qquad (j=1,\ldots,r).$$

The $V_j^{\kappa+1} : \mathbb{C} \to F_2$ are polynomials of degree less than m_j. From (2.8) and (2.10) we conclude that

$$T^{-1}(\lambda) - \sum_{j=1}^{r} (\lambda-\mu)^{-m_j} X_j(\lambda) \otimes V_j^{\kappa+1}(\lambda)$$

$$= \sum_{l=-s+\kappa+1}^{\infty} (\lambda-\mu)^l A_l - \sum_{j=1}^{r} (\lambda-\mu)^{-s+\kappa} (X_j(\lambda)-X_j(\mu)) \otimes v_j ,$$

which shows that (2.7) holds for $\kappa+1$.

The existence of the representation (2.6) is clear with $V_j := V_j^s$. To prove the uniqueness of the V_j we suppose that

$$T^{-1}(\lambda) = \sum_{j=1}^{r} (\lambda-\mu)^{-m_j} X_j(\lambda) \otimes \tilde{V}_j(\lambda)+\tilde{D}(\lambda)$$

where the $\tilde{V}_j$: $\mathbb{C} \to F_2$ are polynomials of degree less than m_j and $\tilde{D}$ is holomorphic in U. It follows that

$$\sum_{j=1}^{r} X_j \otimes (.-\mu)^{-m_j} (V_j-\tilde{V}_j) = \tilde{D}-D$$

which means that the function on the left side is holomorphic in U. By proposition (2.2) we infer that all the $(.-\mu)^{-m_j}(V_j-\tilde{V}_j)$ are holomorphic at μ. Since V_j and $\tilde{V}_j$ are polynomials of degree less than m_j, we obtain $V_j-\tilde{V}_j = 0$ for $j\in\{1,2,...,r\}$.

Next we prove the biorthogonality relationships (2.5): To this purpose we multiply (2.4) by $T(\lambda)$ from the right, which yields

$$(2.11) \qquad id_{E_1} = \sum_{j=1}^{r} (\lambda-\mu)^{-m_j} X_j(\lambda) \otimes (T^*V_j)(\lambda)+(DT)(\lambda),$$

because $T(\lambda)'|_{F_2} = T^*(\lambda)$ $(\lambda\in U)$. From (2.11) the equation

$$X_i(\lambda) = \sum_{j=1}^{r} (\lambda-\mu)^{-m_j} \langle X_i(\lambda),(T^*V_j)(\lambda)\rangle X_j(\lambda)+(DTX_i)(\lambda)$$

is immediate. TX_i, and thus DTX_i has a zero of order m_i at μ, whence the function

$$\sum_{j=1}^{r} X_j\{(.-\mu)^{-m_i}(\delta_{ij}-\langle(.-\mu)^{-m_j}TX_i,V_j\rangle)\}$$

is holomorphic in U. Once more we apply proposition (2.2), this time with $E_2 = \mathbb{C}$, and conclude that the functions

$$(.-\mu)^{-m_i}(\delta_{ij} - \langle(.-\mu)^{-m_j} TX_i,V_j\rangle)$$

$$= (.-\mu)^{-m_j}(\delta_{ij} -\langle(.-\mu)^{-m_i} TX_i,V_j\rangle) \qquad (i,j=1,2,...,r)$$

are holomorphic in μ. Hence, if $1 \leq h \leq m_i$,

$$\delta_{ij}(\cdot-\mu)^{m_i-h} - <(\cdot-\mu)^{-h}TX_i, V_j>$$

has a zero of order not less than m_j at μ, which proves (2.5).

Finally we show that $\{V_1,\ldots,V_r\}$ is a CSRF of T^* at μ with $\nu(V_j) = m_j$: For $h = m_i$ the biorthogonal relationship (2.5) reduces to

$$(2.12) \qquad <\eta_{i,m_i}(\mu), V_j(\mu)> = \delta_{ij} \qquad (i,j=1,2,\ldots,r),$$

whence the vectors $V_1(\mu),\ldots,V_r(\mu)$ are linearly independent and at least different from zero. Since the operator function DT is holomorphic in U, we infer from (2.11) by another application of proposition (2.2) that the functions

$$(\cdot-\mu)^{-m_j}T^*V_j \qquad (j=1,2,\ldots,r)$$

are holomorphic in U, which implies that the V_j are root functions of T^* at μ with $\nu(V_j) \geq m_j$.

Let V be an arbitrary root function of T^* at μ. We multiply (2.4) by $T(\lambda)$ from the left, form the adjoint of both sides of the resulting equation and obtain

$$(2.13) \qquad id_{F_1'} = \sum_{j=1}^{r} (\lambda-\mu)^{-m_j} V_j(\lambda) \otimes (TX_j)(\lambda) + (D'T')(\lambda),$$

which leads to

$$<y, V(\lambda)> = \sum_{j=1}^{r} (\lambda-\mu)^{-m_j} <TX_j(\lambda), V(\lambda)> <y, V_j(\lambda)> + <D(\lambda)y, (T^*V)(\lambda)>$$

for arbitrary $y \in F_1$. Since $D(.)y$ is holomorphic at μ and T^*V vanishes there, the equation

$$<y, V(\mu)> = \sum_{j=1}^{r} <X_j(\mu), ((\cdot-\mu)^{-m_j}T^*V)(\mu)><y, V_j(\mu)>$$

holds. Hence

$$(2.14) \qquad V(\mu) = \sum_{j=1}^{r} <X_j(\mu), ((\cdot-\mu)^{-m_j} T^*V)(\mu)> V_j(\mu),$$

which proves that $\{V_1(\mu),\ldots,V_r(\mu)\}$ is a basis of $N(T^*(\mu))$.

Let $1 \leq k \leq r$ and $V(\mu) \notin \mathrm{span}\{V_1(\mu),\ldots,V_{k-1}(\mu)\}$. From (2.14) we conclude that there is some $j \geq k$ such that

$$<X_j(\mu),((.-\mu)^{-m_j}T^*V)(\mu)> \neq 0,$$

which implies that $\nu(V) \leq m_j \leq m_k \leq \nu(V_k)$. It follows that $\nu(V_k) = m_k$ is the maximum of all $\nu(V)$ where V varies in the set of all root functions of T^* at μ, such that

$$V(\mu) \notin \text{span}\{V_1(\mu),\dots,V_{k-1}(\mu)\}.$$

We supplement lemma (2.3) by the following

(2.15) Remark. Let $\mu,\tilde{\mu}\in U$ and $\mu \neq \tilde{\mu}$. Suppose that X is a root function of T at μ, and that V is a root function of T^* at $\tilde{\mu}$. Set $\eta_h(\lambda) = (\lambda-\mu)^{-h}(TX)(\lambda)$ for $\lambda\in U$ and $1 \leq h \leq \nu(X)$. Then

$$\frac{d^l}{d\lambda^l} <\eta_h,V>(\tilde{\mu}) = 0 \tag{2.16}$$

for all $1 \leq h \leq \nu(X)$ and $0 \leq l \leq \nu(V) - 1$.

Proof. Since $(.-\mu)^{-h}$ is holomorphic at $\tilde{\mu}$, the function

$$<\eta_h,V> = <(.-\mu)^{-h}X,T^*V>$$

has a zero of order not less than $\nu(V)$ there.

A similar argument yields the

(2.17) Remark. Let $Z\in H(U,E_1)$. In (2.5) we may substitute η_{ih} by

$$(.-\mu)^{-h}T(X_i+(.-\mu)^hZ)$$

and in (2.16) η_h by

$$(.-\mu)^{-h}T(X+(.-\mu)^hZ).$$

The following result is an immediate consequence of lemma (2.3) and the remarks (2.15) and (2.17).

(2.18) Theorem. *Assume that* $\sigma(T)$ *is a discrete subset of* U, *that* T^{-1} *and* T^{*-1} *are holomorphic in* $U\setminus\sigma(T)$, *and that the singularities of* T^{-1} *are poles. For each* $\mu\in\sigma(T)$ *let*

$\{x_{l,\mu}^{(j)} : 1 \leq j \leq r(\mu),\ 0 \leq l \leq m_j(\mu)-1\}$ *be a* CSEAV *of* T *at* μ.

Then there are systems

$$(2.19)\qquad \{v^{(j)}_{l,\mu} : 1 \le j \le r(\mu),\ 0 \le l \le m_j(\mu) - 1\} \subset F_2$$

such that

$$(2.20)\qquad SP(T^{-1},\mu)(\lambda) = \sum_{j=1}^{r(\mu)} \sum_{k=1}^{m_j(\mu)} (\lambda-\mu)^{-k} \sum_{l=0}^{m_j(\mu)-k} x^{(j)}_{l,\mu} \otimes v^{(j)}_{m_j(\mu)-l-k,\mu},$$

where $SP(T^{-1},\mu)$ *denotes the singular part of* T^{-1} *at* μ. (2.19) *is a* CSEAV *of* T^* *at* μ *and is uniquely determined by the* CSEAV *of* T *at* μ. *The biorthogonal relationships*

$$(2.21)\qquad \sum_{n=0}^{l} \frac{1}{n!} \langle \eta^{(n)}_{i,h,\mu_1}(\mu_2), v^{(j)}_{l-n,\mu_2}\rangle = \delta_{\mu_1,\mu_2}\delta_{ij}\delta_{m_i(\mu_1)-h,l}$$

$$(\mu_1,\mu_2\in\sigma(T), 1\le i\le r(\mu_1), 1\le h\le m_i(\mu_1), 1\le j\le r(\mu_2), 0\le l\le m_j(\mu_2)-1)$$

hold where

$$\eta_{i,h,\mu_1}(\lambda) := T(\lambda)\sum_{l=0}^{h-1}(\lambda-\mu_1)^{l-h} x^{(i)}_{l,\mu_1} .$$

We only have to prove (2.20). We omit the index μ, use the notations of section 1 and conclude from (2.4)

$$SP(T^{-1},\mu) = SP(\sum_{j=1}^{r}(.-\mu)^{-m_j} X_j\otimes V_j,\mu)$$

$$= SP(\sum_{j=1}^{r}(.-\mu)^{-m_j}(\sum_{l=0}^{m_j-1}(.-\mu)^l x^{(j)}_l) \otimes (\sum_{k=0}^{m_j-1}(.-\mu)^k v^{(j)}_k),\mu)$$

$$= \sum_{j=1}^{r}\sum_{l=0}^{m_j-1}\sum_{k=-m_j+1}^{-1}(.-\mu)^k x^{(j)}_l \otimes v^{(j)}_{k+m_j-l}$$

$$= \sum_{j=1}^{r}\sum_{k=1}^{m_j}(.-\mu)^{-k}\sum_{l=0}^{m_j-k} x^{(j)}_l \otimes v_{m_j-k-l}.$$

3. An application.

In this section we study an application to eigenvalue problems for differential equations in the complex domain in which all the assumptions made in theorem (2.18), are fulfilled in a natural way. We would like to point out that the eigenvalue problems considered here are closely related to the theory of special functions, and that the theorem (2.18) is the starting-point for a large number of biorthogonal expansions of holomorphic functions into series of special functions, cf. Krimmer [10].

Let S be a simply connected domain in $\mathbb{C}$ which we assume to be 2π-periodic, i.e. $z\in S$ iff $z+2\pi\in S$. Let $\nu\in\mathbb{C}$ and define

$$P_\nu := \{f\in H(S) : \forall z\in S \quad f(z+2\pi) = e^{2\pi i\nu}f(z)\}.$$

P_ν is a closed subspace of the Fréchet space H(S), and hence also a Fréchet space. For $f\in P_\nu$ and $g\in P_{-\nu}$ we set

(3.1) $$\langle f,g\rangle := \int_\gamma f(z)g(z)dz$$

where γ is a rectifiable arc in S from a fixed z_0 to $z_0+2\pi$. Obviously the integral is independent of the choice of z_0 and the arc γ.

(3.2) Proposition. $(P_\nu, P_{-\nu})$ is a dual pair with respect to $\langle\ ,\ \rangle$, which is continuous in both components.

We only have to show that the bilinear form (3.1) is definite. For this purpose we prove:

(3.3) The set $\hat{S} := \{e^{iz} : z\in S\}$ is a domain of connectivity 2.

Proof. $\hat{S}$ is the image of a connected set under a continuous mapping, and hence $\hat{S}$ is connected. The arc $\hat{\gamma} : [0,1] \to \hat{S}$ defined by $\hat{\gamma}(t)=e^{i\gamma(t)}$ is a closed rectifiable curve, and the winding number of $\hat{\gamma}$ with respect to 0 is 1. As $0\notin\hat{S}$ we infer that $\hat{S}$ is not simply connected.

Suppose $\hat{S}$ is not a domain of connectivity 2. Then there are three pairwise disjoint nonempty compact subsets A_1,A_2,A_3 of $\overline{\mathbb{C}}$, such that $0\in A_1,\infty\in A_2$ and $\overline{\mathbb{C}}\setminus\hat{S} = A_1\cup A_2\cup A_3$. The sets $B_1 = \{z\in\mathbb{C}: e^{iz}\in A_1\cup A_2\}$ and $B_3 = \{z\in\mathbb{C}: e^{iz}\in A_3\}$ are closed disjoint subsets of $\mathbb{C}$ with $B_1\cup B_3 = \mathbb{C}\setminus S$. As $B_1\cup B_3\cup\{\infty\} = \overline{\mathbb{C}}\setminus S$ is connected, B_3 cannot have a bounded component. As A_3 is bounded away from 0 and ∞, there are two points $z_1,z_2\in S$ such that $|z_1| < \inf\{|\operatorname{Im} z| : z\in B_3\}$ and $|z_2| > \sup\{|\operatorname{Im} z| : z\in B_3\}$. We choose a Jordan arc $\gamma_1: [0,1] \to \mathbb{C}\setminus B_3$ which meets the lines $\operatorname{Im} z = \operatorname{Im} z_i$ $(i=1,2)$ only in the endpoints z_1 and z_2. Let $k\in\mathbb{N}$; $\gamma_2(t) = \gamma_1(1-t)+2k\pi$ defines an arc $\gamma_2 : [0,1] \to \mathbb{C}\setminus B_3$. If k is large enough, then γ_1 and γ_2 together with the lines $\overline{z_2,z_2+2k\pi}$ and $\overline{z_1+2k\pi,z_1}$ form a Jordan curve in $\mathbb{C}\setminus B_3$ which has points of B_3 in its interior, a contradiction to the fact that B_3 does not have bounded components.

It is sufficient to prove the definiteness of (3.1) in the case $\nu=0$. Let $f\in P_0$ and suppose that $\langle f,g\rangle = 0$ for all $g\in P_0$. It follows that

$$\int_{\hat{\gamma}} \hat{f}(w)\hat{g}(w)dw = 0$$

for all $\hat{g}\in H(\hat{S})$, where $\hat{f}(e^{iz}) = f(z)$ for all $z\in S$ and $\hat{\gamma}$ has been defined in the proof of (3.3). Since $\hat{S}$ is a domain of connectivity 2 we can assume without loss of generality that $\hat{S}$ is an annulus around 0 (cf. Ahlfors [1], p. 247). We conclude that

$$\int_{\hat{\gamma}} \hat{f}(w)w^n dw = 0 \quad (n\in\mathbb{Z}),$$

whence $\hat{f} = 0$ as $\hat{f}\in H(\hat{S})$.

Let $n\in\mathbb{N}\setminus\{0\}$ and $p_j\in H(S\times\mathbb{C})$ for $j=0,\ldots,n-1$. We assume that the functions $p_j(.,\lambda)$ are 2π-periodic for each $\lambda\in\mathbb{C}$. For $f\in P_\nu$ we set

$$(T(\lambda)f)(z) := f^{(n)}(z) + \sum_{j=0}^{n-1} p_j(z,\lambda)f^{(j)}(z) \quad (z\in S)$$

and state that $T\in H(\mathbb{C},L(P_\nu,P_\nu))$. It is well-known that $\sigma(T) = \sigma_p(T)$ and that either $\sigma(T) = \mathbb{C}$ or $\sigma(T)$ is a discrete subset of $\mathbb{C}$. For the following we assume that $\sigma(T) \neq \mathbb{C}$.

Let $Y(z,\lambda)$ be a fundamental matrix of the differential equation $T(\lambda)f = 0$ with $Y(z_0,\lambda) = I$. We know that $Y\in H(S\times\mathbb{C},M_n(\mathbb{C}))$. It follows, cf. Krimmer [10], that $\lambda\in\sigma(T)$ iff the matrix

$$M_Y(\lambda) := e^{2\pi i\nu}I - Y(z_0+2\pi,\lambda)$$

is not invertible, and that

$$(3.4) \qquad (T^{-1}(\lambda)g)(z) := \int_z^{z+2\pi} G_{1n}(z,\xi;\lambda)g(\xi)d\xi \quad (z\in S)$$

for all $\lambda\in\rho(T)$. In (3.4) $G_{1n}(z,\xi;\lambda)$ denotes the element with index $(1,n)$ of the "Green matrix"

$$G(z,\xi;\lambda) = Y(z+2\pi,\lambda)M_Y^{-1}(\lambda)Y(\xi,\lambda)^{-1},$$

and the integration is carried out along a rectifiable arc in S. According to (3.4), T^{-1} belongs to $H(\rho(T),L(P_\nu,P_\nu))$ and has poles at all points of $\sigma(T)$.

The operator $T^*(\lambda)$ defined by

$$(T^*(\lambda)g)(z) := (-1)^n g^{(n)}(z) + \sum_{j=0}^{n-1} (-1)^j (p_j(.,\lambda)g)^{(j)}(z) \quad (g\in P_{-\nu}, z\in S)$$

is the "adjoint" operator of $T(\lambda)$ with respect to the dual pairs $(E_1,E_2) = (F_1,F_2) = (P_\nu,P_{-\nu})$. It is easy to show that $\sigma(T^*) = \sigma(T)$. Since T^* has the same form as T, we have $T^*\in H(\mathbb{C},L(P_{-\nu},P_{-\nu}))$ and $T^{*-1}\in H(\rho(T),L(P_{-\nu},P_{-\nu}))$. Thus all assumptions of theorem (2.18) are fulfilled in this case.

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THE THEOREM OF BOCHNER-SCHWARTZ-GODEMENT FOR GENERALISED GELFAND PAIRS

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Introduction

Fourier analysis on the circle T is essentially concerned with the decomposition

(1) $$L^2(T) = \sum_n{}^{\oplus} [e^{in\theta}]$$

of L^2 into minimal translation invariant subspaces. Similarly, Fourier analysis on $\mathbb{R}$ is concerned with the decomposition

(2) $$L^2(\mathbb{R}) = \int^{\oplus} [e^{2\pi i\lambda\cdot}]d\lambda$$

into minimal translation invariant subspaces, not of L^2, which has no such minimal invariant subspaces, but of an appropriate larger space, e.g. $\mathcal{S}'(\mathbb{R})$.

More generally, let X be a C^∞-manifold on which a Lie group G acts transitively from the left (then if $p \in X$ and $G_0 = \{g \in G : gp = p\}$ one may as usual identify X with G/G_0). The group G also acts naturally, by 'translation' on the space $\mathcal{D}(X) = C_c^\infty(X)$ and on the space of distributions $\mathcal{D}'(X)$.

If the space X possesses a G-invariant positive Radon measure dx, the space $L^2(X,dx)$ is a Hilbert space continuously embedded in $\mathcal{D}'(X)$

$$L^2(X) \hookrightarrow \mathcal{D}'(X)$$

and invariant under translation (i.e. the set L^2 is invariant and the restrictions of the translation operators to L^2 are unitary).

Consider more generally the set $\mathrm{Hilb}_G(\mathcal{D}'(X))$ of all G-invariant Hilbert subspaces $H \hookrightarrow \mathcal{D}'(X)$. We shall be concerned with the decomposition of such a space into minimal (i.e. irreducible) invariant subspaces of $\mathcal{D}'(X)$:

(3) $$H = \int^{\oplus} H_\lambda \, dm(\lambda)$$

analogous to (1) and (2).

To make this more precise we make use of the fact that the set $\mathrm{Hilb}_G(\mathcal{D}'(X))$ is in bijective correspondance with the set Γ_G of distributions $K \in \mathcal{D}'(X \times X)$ which are of positive type :

$$\langle K , \phi \otimes \overline{\phi} \rangle \geq 0 \qquad \forall \phi \in \mathcal{D}(X)$$

and which are invariant under the diagonal action of $G : (x,y) \to (gx,gy)$. In this correspondance the distribtion K associated with the space H is the Schwartz reproducing kernel of H. The minimal invariant spaces correspond to the extreme generators of the convex cone Γ_G.

The existence of decompositons such as (3) has been known for some time. In general there are infinitely many ways to do such a decomposition (e.g. if $X = G$ acts on itself from the left and is noncommutative).

The success of classical Fourier analysis associated with the decomposition (2), however, is mainly due to the fact that the functions $e^{2\pi i\lambda x}$ are eigenfunctions of the operators $\mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ which commute with translations. This is closely related to the fact that in (2) the representing measure is uniquely determined, i.e. only one decomposition is possible.

In the case of a general homogeneous space we shall also be concerned with the situation in which the decomposition is unique, or equivalently, where the decomposition

$$(4) \qquad K = \int K_\lambda \, dm(\lambda)$$

of a G-invariant kernel of positive type into extreme kernels is unique. By making use of the theory of integral representation one can show that this uniqueness is equivalent to Γ_G being a lattice cone.

In the classical examples of (4) associated with the names of Herglotz, Bochner, Weil, Raikov, Schwartz, Godement and others, one has this type of uniqueness. In these cases the stability subgroup G_0 is compact. If G_0 is compact it can be shown that Γ_G is a lattice if and only if (G,G_0) is a Gelfand pair, i.e. the convolution algebra $L^1(G_0 \backslash G/G_0)$ of L^1 functions on G, bi-invariant under G_0, is commutative. The extreme kernels then correspond to the positive definite spherical functions.

In this talk we shall also discuss recent progress in the case where the stability subgroup G_0 is not compact, in which case we shall propose several equivalent definitions of 'Gelfand pair' to replace the classical definition which becomes inoperative ($L^1(G_0 \backslash G/G_0)$ being (0)). We shall also show that the classical sufficient condition which implies that (G,G_0) is a Gelfand pair, namely the existence of an appropriate symmetry, can, in modified form, still be used in the general case.

1. Hilbert subspaces of $\mathcal{D}'(X)$

We recall the basic facts from L. Schwartz's theory of Hilbert subspaces of the space $\mathcal{D}'(X)$, where X is a C^∞-manifold [8][9].

A Hilbert subspace of $\mathcal{D}'(X)$ is a linear subspace H of $\mathcal{D}'(X)$ equipped with an inner product which makes it into a Hilbert space, and such that the inclusion

$$H \underset{j}{\hookrightarrow} \mathcal{D}'(X)$$

is continuous.

The sum $H_1 + H_2$ of two Hilbert subspaces itself is a Hilbert subspace if it is equipped with the norm for which the map $s : H_1 \times H_2 \to H_1 + H_2$ is a partial isometry, i.e. the quotient map $H_1 \times H_2/_{\ker(s)} \to H_1 + H_2$ is an isometry. (Here $H_1 \times H_2$ has the usual Hilbert space norm). If s is one-to-one, hence an isometric isomorphism, the sum is said to be direct, and we write $H_1 \oplus H_2$. This occurs if and only if $H_1 \cap H_2 = (0)$.

For any Hilbert subspace $H \subset \mathcal{D}'(X)$ and any number $\alpha \geq 0$, one defines the Hilbert subspace αH to be (0) if $\alpha = 0$, and to be the linear space H with the inner product

$$(h_1|h_2)_{\alpha H} = \frac{1}{\alpha}(h_1|h_2)_H$$

if $\alpha > 0$.

An order relation is defined on the set of Hilbert subspaces as follows:

$$H_1 \leq H_2 \Leftrightarrow B_1 \subset B_2$$

where B_i is the closed unit ball of H_i.

Finally, given any Hilbert space H and a continuous linear map $u : H \to \mathcal{D}'(X)$ the image space u(H) becomes a Hilbert subspace if it is equipped with the norm which makes u a partial isometry, i.e. the quotient map an isometry. The image u(H) is always considered to be a Hilbert space in this way.
(Examples: $H_1 + H_2 = s(H_1 \times H_2)$, αH is the image of H under the map $h \to \alpha^{1/2}h$). In particular, if $u : \mathcal{D}'(X) \to \mathcal{D}'(X)$ is a continuous linear map and H is a Hilbert subspace of $\mathcal{D}'$, u(H) is another Hilbert subspace. If the restriction $u/_H$ is one-to-one it is an isometric isomorphism of H onto u(H).

We denote Hilb($\mathcal{D}'$) the set of all Hilbert subspaces of $\mathcal{D}'(X)$.

The Schwartz reproducing kernel of a Hilbert subspace $H \underset{j}{\hookrightarrow} \mathcal{D}'$ is the distribution $K \in \mathcal{D}'(X \times X)$, defined, via the kernel theorem, by the equation

(5) $$K(\varphi \otimes \overline{\psi}) = (j^*\varphi|j^*\psi)_H$$

j^* being the adjoint map defined by the equation

(6) $$(h|j^*\varphi) = \langle jh, \varphi \rangle$$

where for convenience we have put

(7) $$\langle f, \varphi \rangle = f(\overline{\varphi})$$

when $f \in \mathcal{D}'(X)$, $\varphi \in \mathcal{D}(X)$. The distribution K is a kernel of positive type :

(8) $$K(\varphi \otimes \bar{\varphi}) \geq 0 \qquad \forall \varphi \in \mathcal{D}(X).$$

We denote Γ the set of all distributions $K \in \mathcal{D}'(X \times X)$ which satisfy (8). It is a closed convex cone of $\mathcal{D}'(X \times X)$, which is proper i.e. $\Gamma \cap -\Gamma = (0)$.

We are now in a position to state L. Schwartz's fundamental result concerning Hilbert subspacesand their reproducing kernels :

Proposition 1 The correspondence $H \to K$ which associates with each Hilbert subspace of $\mathcal{D}'(X)$ its reproducing kernel, is a bijection between $\mathrm{Hilb}(\mathcal{D}')$ and Γ. Moreover :

1. If K_i is the reproducing kernel of H_i, $i = 1,2$, $K_1 + K_2$ is the reproducing kernel of $H_1 + H_2$.
2. If K is the reproducing kernel of H, αK is the reproducing kernel of αH.
3. $H_1 \leq H_2$ if and only if $K_1 \leq K_2$ (i.e. $K_2 - K_1 \in \Gamma$)
4. If $u : \mathcal{D}'(X) \to \mathcal{D}'(X)$ is a continuous linear map and H has kernel K, the space $u(H)$ has the kernel $u(K)$ defined by

(9) $$u(K)(\varphi \otimes \psi) = K(u^*\varphi \otimes u^*\psi)$$

where $u^* : \mathcal{D}(X) \to \mathcal{D}(X)$ is defined by

(10) $$\langle uf, \varphi \rangle = \langle f, u^*\varphi \rangle$$

Corollary 2 $u(H) = H \Leftrightarrow K(u^*\varphi \otimes u^*\psi) = K(\varphi \otimes \psi) \qquad \forall \varphi, \psi \in \mathcal{D}$

2. Integrals of Hilbert subspaces

Let us now define integrals of Hilbert subspaces, in part following [8]. Let S be a topological Hausdorff space, and let m be a Radon measure on S, (i.e. a locally finite inner regular Borel measure).

A family $(H_s)_{s \in S}$ of Hilbert subspaces of $\mathcal{D}'(X)$ is said to be m-measurable if the corresponding map $s \to K_s$ is μ-measurable, equivalently : $s \to K_s(\varphi \otimes \psi)$ is m-measurable for all $\varphi, \psi \in \mathcal{D}(X)$ (cf. [15] thm 1).

Given an m-measurable family of Hilbert subspaces $(H_s)_{s \in S}$, a field $f(\,.\,) = (f(s))_{s \in S} \in \prod_{s \in S} H_s$ is said to be measurable if the vector function $s \to f(s) \in \mathcal{D}'(X)$ is m-measurable, equivalently : weakly m-measurable.

It can be shown that the map $s \to \|f(s)\|_s$ is measurable and that the space $L^2 = L^2(m, \{H_s\}, \mathcal{D}')$ of equivalence classes of square integrable fields, equipped with the norm

(11) $$\|f(\,.\,)\| = \left(\int \|f(s)\|_s^2 \, dm(s)\right)^{1/2},$$

is a Hilbert space.

In fact, $(H_s)_{s \in S}$, together with the family of m - measurable fields is a measurable family of Hilbert spaces in the sense of von Neumann and Dixmier [2].

The family $(H_s)_{s \in S}$ is said to be m - summable if it is measurable and if

$$\int K_s(\varphi \otimes \overline{\varphi})\,dm(s) < +\infty$$

for all $\varphi \in D$. There then exists a distribution $K \in \Gamma$ such that

$$K(\varphi \otimes \psi) = \int K_s(\varphi \otimes \psi)\,dm(s) \qquad \forall \varphi, \psi \in D\,.$$

If H is the Hilbert space whose reproducing kernel is K, we write symbolically

(12) $$H = \int H_s\,dm(s)\,.$$

Similarly we define for every measurable subset $A \subset S$,

$$H_A = \int_A H_s\,dm(s)\,.$$

Finally, a function $f : S \to D'(X)$ is said to be m - summable if the integral $\int f(s)dm(s)$ exists in $D'(X)$, equivalently $s \to \langle f(s), \varphi \rangle$ is m - summable for every $\varphi \in D(X)$ (cf. [15]).

Proposition 3 Let $(H_s)_{s \in S}$ be an m - summable family of Hilbert subspaces of $D'(X)$, and let

$$H = \int H_s\,dm(s)$$

Then we have :

1. Every square integrable field is an m - summable map $f : S \to D'$.
2. If $\Phi[f(\,.\,)] = \int f(s)dm(s)$, the map $\Phi : L^2 \to D'$ is linear and continuous.
3. $H = \Phi(L^2)$.

For the proof we refer to Schwartz [8].

The integral (12) is said to be direct, and written

(13) $$H = \int^{\oplus} H_s\,dm(s)$$

if the map $\Phi : L^2 \to H$ is one-to-one, hence an isometric isomorphism. This happens if and only if $H_A \cap H_B = 0$ for any two disjoint Borel sets, A and B (disjoint compact sets is not enough).

We shall describe the situation more fully in this case :

Proposition 4 If $H = \int^{\oplus} H_s\,dm(s)$ we have the following :

1. Every element $f \in H$ has a unique decomposition $f(\cdot) \in L^2$ such that, in $D'(X)$,

(14) $$f = \int f(s)dm(s)$$

and every $f(\cdot) \in L^2$ gives rise in this way to an element $f \in H$.

2. One has the 'continuous Pythagoras theorem' :

(15) $$||f||^2 = \int ||f(s)||_s^2 \, dm(s).$$

3. The space H_A is a closed subspace of H, its norm being inherited from H. The orthogonal projection P_A from H onto H_A is given by :

(16) $$P_A f = \int_A f(s) dm(s).$$

For $A \cap B = \emptyset$ H_A and H_B are orthogonal subspaces of H.

The proof of this is quite obvious if one realizes that H_A is the image under Φ of the space of fields in L^2, vanishing off A.

Remark The von Neumann algebra A generated by the projections P_A is precisely equal to the set of diagonal operators T_φ, for $\varphi \in L^\infty(m)$

(17) $$T_\varphi f = \int \varphi(t) f(t) dm(t).$$

Thus the decomposition (13) is called the diagonalisation of the von Neumann algebra A. In a certain precise sense it depends only on H and A, and not on S and m (cf. [14]).

Proposition 5 If $H = \int^\oplus H_s dm(s)$, the map $A \to P_A$ is an m-continuous spectral measure. Conversely, if H is a Hilbert subspace of $D'(X)$, and P is an m-continuous spectral measure there exists an essentially unique m-summable family of Hilbert subspaces $(H_s)_{s\in S}$ such that, if $H_A = P_A H$, one has $H_A = \int_A H_s \, dm(s)$. Moreover the integral is direct :

$$H = \int^\oplus H_s \, dm(s).$$

This can be proved in many ways. The best is to regard it as a Radon Nikodym theorem for the vector valued measure $A \to H_A \in \mathrm{Hilb}(D')$ (or $A \to K_A \in \Gamma$). The last statement of the proposition is obvious since $H_A \cap H_B = (0)$ for $A \cap B = \emptyset$.

Corollary 6 Every commutative von Neumann algebra, acting on a Hilbert subspace of $D'(X)$, may be diagonalised in $\mathrm{Hilb}(D')$.

Corollary 7 If Γ_0 is a closed convex cone in Γ, $\mathrm{Hilb}_0(D')$ the corresponding set of Hilbert subspaces we have

$$H_A \in \mathrm{Hilb}_0(D') \forall A \Leftrightarrow H_s \in \mathrm{Hilb}_0(D') \text{ m a.a.s}$$

3. G-invariant spaces

Let us now assume $X = G/G_0$ where G is a Lie group, G_0 a closed subgroup. Then G acts on $D(X)$ by $u(g)\varphi(x) = \varphi(g^{-1}x)$, and on $D'(X)$ by

$$\langle u(g)f,\varphi\rangle = \langle f,u(g^{-1})\varphi\rangle$$

If we assume, as we shall, that X has a G-invariant Radon measure dx, the spaces $\mathcal{D}$ and $L^2(X,dx)$ are considered as subspaces of $\mathcal{D}'(X)$:

$$\mathcal{D}(X) \hookrightarrow L^2(X) \hookrightarrow L^1_{loc}(X) \hookrightarrow \mathcal{D}'(X)$$

and the action of G on $\mathcal{D}$ or L^2 coincides with the restriction to those spaces of the action on $\mathcal{D}'$.

We denote $Hilb_G(\mathcal{D}')$ the set of all Hilbert subspaces of $\mathcal{D}'(X)$ which are invariant under G, i.e. for which

(18) $\qquad u(g)H = H \qquad \forall g \in G.$

For each $H \in Hilb_G(\mathcal{D}')$ the map $g \to u(g)_{/H}$ is a continuous unitary representation of G in the usual sense. We denote $G|H = \{u(g)/_H, g \in G\}$

We denote Γ_G the closed convex subcone of Γ composed of the kernels $K \in \Gamma$ for which

(19) $\qquad K(u(g)\varphi \otimes u(g)\psi) = K(\varphi \otimes \psi) \qquad \forall g \in G$

i.e. which are invariant under the diagonal action $(x,y) \to (gx,gy)$ on $X \times X$. Then by Corollary 2, H is G-invariant if and only if its reproducing kernel belongs to Γ_G.

Proposition 8 The space H is minimal invariant (i.e. it contains no closed invariant subspaces) if and only if its kernel K lies on an extreme ray of the cone Γ_G, i.e. :

$$0 \le K' \le K,\ K' \in \Gamma_G \ \Rightarrow\ K' = \alpha K.$$

For the proof we refer to [9] p.71, or [13].

Now it is a fact that there are actually plenty of minimal invariant spaces (Γ_G is the closed convex hull of its extreme rays).

This can be expressed more precisely as follows : let $ext(\Gamma_G)$ be the union of the extreme rays of Γ_G. Let S_0 be a section of $ext(\Gamma_G)$ i.e. a set not containing 0, and having precisely one point on each extreme ray. A section is said to be admissible if the function equal to 1 on S_0 and homogeneous of degree 1, is universally measurable. Such admissible sections always exist, in practice one can often find Souslin sections, which are always admissible. An admissible parametrisation of $ext(\Gamma_G)$ is a continuous one-to-one map $s \to K_s \in ext(\Gamma_G)$ such that the image is an admissible section and the inverse map is universally measurable (e.g. $S = S_0$ with the identity).

Then we have

Proposition 9 Let $s \to K_s$ be an admissible parametrisation of $\mathrm{ext}(\Gamma_G)$, let H_s be the Hilbert subspace corresponding to K_s. Then, for every $H \in \mathrm{Hilb}_G(\mathcal{D}')$ there exists a Radon measure m on S such that

$$H = \int^{\oplus} H_S \, dm(s). \tag{19}$$

This result, except for the fixed parametrisation independent of H, has been obtained by L. Schwartz [10] and K. Maurin [7].

The proof of proposition 9 is obtained by diagonalising a maximal commutative C* algebra commuting with the action of G in H, (corollary 6 and 7, with $\Gamma_0 = \Gamma_G$). Then by Mautmer's theorem the G - invariant component spaces are a.a irreducible. The fixed parametrisation can then be obtained by the techniques of [16].

We shall need proposition 9 only for the precise formulation of the results in the next paragraph where we discuss the uniqueness of the representing measure m in (19).

4. Main results

Theorem A The following are equivalent :

1. If H_1 and H_2 are minimal G - invariant Hilbert subspaces of $\mathcal{D}'(X)$ which are not proportional (i.e. which differ as linear subspaces) the irreducible representations $G|H_1$ and $G|H_2$ are inequivalent.
2. If H is any G - invariant Hilbert subspace of $\mathcal{D}'(X)$, the commutant $(G|H)'$ is abelian.
3. The convex cone Γ_G of all G - invariant positive kernels on X x X is a lattice cone.
4. For every $K \in \Gamma_G$ there exists a unique Radon measure m on S such that

$$K = \int K_s \, dm(s).$$

5. For every G - invariant Hilbert subspace $H \hookrightarrow \mathcal{D}'(X)$ there exists a unique Radon measure m on S such that

$$H = \int^{\oplus} H_s \, dm(s).$$

5'. In particular, there is a unique Radon measure ds on S such that

$$L^2(X) = \int^{\oplus} H_s \, ds .$$

Definition If the properties 1. to 5. are satisfied we say that (G,X) is multiplicity free.

Remark On an earlier occasion we have shown that conditions 1. to 5. are equivalent under the (superfluous) assumption that G be of type I.

Before giving a proof of theorem A we state some related results.

In the next statement we let G' stand for the algebra of all continuous linear operators $v : \mathcal{D}'(X) \to \mathcal{D}'(X)$ which commute with the action of G :

$$vu(g) = u(g)v \qquad \forall g \in G.$$

Theorem B Let (G,X) be multiplicity free. Then we have the following:

6. Every minimal G - invariant Hilbert subspace $H \hookrightarrow \mathcal{D}'(X)$ is an eigenspace for the operators $v \in G'$

$$vf = \lambda f \qquad \forall f \in H$$

where $\lambda = \lambda(v,H)$.

7. The algebra G' is commutative. In particular :

7'. Any two invariant differential operators commute.

Proof of 6. and 7. Let $v \in G'$ and let H be a minimal invariant space. If $v(H) = (0)$ 6. is obvious with $\lambda = 0$. If $v(H) \neq 0$ we have $u(g)v(H) = vu(g)H = v(H)$, and so v(H) is invariant, v being intertwining between the action of G in H and in v(H). H being irreducible the kernel of $v/_H$ is (0) and so $v : H \longrightarrow v(H)$ is an isomorphism. By condition 1. $v(H) = \alpha H$ for some α. But then by Schur's lemma $v/_H$ must be a multiple of the identity. Let $f \in L^2$. Then in $\mathcal{D}'$ we have

$$f = \int f(s)ds$$

with $f(s) \in H_s$.

If $v,w \in G'$ we have in $\mathcal{D}'$

$$vf = \int v[f(s)]ds = \int \lambda(v,s)f(s)ds$$

where $\lambda(v,s)$ is the eigenvalue of $v/_{H_s}$.

$$\begin{aligned} wvf &= \int w\lambda(v,s)f(s)ds \\ &= \int \lambda(v,s)w[f(s)]ds \\ &= \int \lambda(v,s)\lambda(w,s)f(s)ds \\ &= vwf. \end{aligned}$$

Now L^2 being a dense subspace of $\mathcal{D}'$, $wv = vw$.

Remark If the stability subgroup G_0 is compact, it can be shown that conversely 6. implies 1. to 5. and that 7. implies 1. to 5. If G is connected even 7.' implies 1. to 5. This explains a statement in the introduction. (cf. [11]).

Theorem C Let (G,X) be multiplicity free. Let $H \subset \mathcal{D}'(X)$ be any G - invariant Hilbert subspace (e.g. $L^2(X,dx)$).

8. Let (T,D_T) be any closed operator in H commuting with the action of G. Then T is normal.
9. Let (T,D_T) be any densely defined symmetric operator in H, commuting with the action of G. Then T is essentially selfadjoint.
10. Let T_1 and T_2 be as in 8. then T_1 and T_2 strongly commute.

Example If X admits a G - invariant pseudo Riemannian structure, the Laplace Beltrami operator Δ, defined on $\mathcal{D}(X)$ is essentially selfadjoint in $L^2(X)$.

Proof of Theorem C 8.: The operators $B = (I + T^*T)^{-1}$ and $C = T(I + T^*T)^{-1}$ are bounded operators commuting with G. Thus $(G|H)'$ being an abelian * algebra, they are normal, they commute, and have commuting spectral resolutions. From this it is easy to deduce that T has a spectral resolution, i.e. T is normal.
9.: The closure of T is a normal symmetric operator, hence selfadjoint. Alternatively: The Cayley transform U of the closure of T is a partial isometry such that the operators UU^* and U^*U are the projections on the defect spaces $\mathrm{Ker}(T^* \pm iI)^{\perp}$. But since U belongs to $(G|H)'$, $UU^* = U^*U$, hence $\mathrm{Ker}(T^* + i\,I) = \mathrm{Ker}(T^* - i\,I) = (0)$.
10.: The spectral resolutions of T_1 and T_2 belongs to $(G|H)'$, hence they commute.

The following three theorems allow one to show that (G,X) is multiplicity free in certain cases :

Theorem D Let G_1 and G_2 be two groups acting differentiably on X, such that G_1 is a subgroup of G_2. Then if (G_1,X) is multiplicity free, so is (G_2,X).

Proof Let H be G_2 invariant. Then H is G_1 invariant and $(G_2|H)' \subset (G_1|H)'$. Thus the smaller commutant is also abelian.

Examples
a) Let $X = \mathbb{R}^n$ $G_1 = \mathbb{R}^n$ acting by translation. Then G_1 being abelian the minimal invariant spaces are one-dimensional and so they are, up to a factor, the one dimensional spaces spanned by the characters $x \to e^{ix\cdot y}$. Since two different characters give inequivalent representations (G_1,X) is multiplicity free. The invariant operators are the operators of convolution with a distribution of compact support, in particular the differential operators with constant coefficients, and the $e^{ix\cdot y}$ are of course common eigenvectors for those.

b) Let G_2 be the Poincaré group i.e. the group of transformations of $\mathbb{R}^n$ of the form $x \longrightarrow \Lambda x + b$, with Λ a Lorentz transformation.

Then, by theorem D, (G_2, X) is also multiplicity free. The minimal invariant spaces yield inequivalent representations of the Poincaré group. They are eigenspaces of the Klein - Gordon operator, (the Laplace Beltrami operator for the Einstein metric), a fact extensively remarked upon by L. Schwartz [9].

c) More generally, if G is a semi-direct product of an abelian group N and a group of Haar measure preserving automorphisms of N, then (G,N) is multiplicity free. For more details we refer to F. Klamer [5].

Theorem E Let $J : \mathcal{D}'(X) \to \mathcal{D}'(X)$ be an anti-automorphism. If $JH = H$ (i.e. $J/_H$ anti unitary) for all G - invariant H (or minimal G - invariant H) the pair (G,X) is multiplicity free.

Proof If the minimal invariant spaces are invariant under J, so are the others, for instance by proposition 9. Let H be G - invariant, and let $\mathcal{A} = (G|H)'$. The restriction $\underline{J} = J/_H$ is an anti-unitary operator in H by hypothesis. If A is a positive operator in $\mathcal{A}$ the kernel K_1 defined by

$$K_1(\varphi \otimes \overline{\psi}) = (Aj^*\varphi | j^*\psi)$$

is the reproducing kernel of a G - invariant space $H_1 \hookrightarrow H$. Since $JH_1 = H_1$ it follows that $\underline{J}\, A\, \underline{J}^{-1} = A$. Thus for any $A \in \mathcal{A}$ we have

$$A^* = \underline{J}\, A\, \underline{J}^{-1},$$

the operator $\underline{J}$ being antilinear. Thus if $A, B \in \mathcal{A}$, we have $(AB)^* = A^*B^*$, the 'wrong' equation, i.e. $\mathcal{A}$ is commutative. (This of course has a close connection with the argument of Lichnerowicz in [6]).

Examples

a) Let $X = G_0$ be a unimodular Lie group, $G = G_0 \times G_0$ acting from the left and right

$$(g_1, g_2)x = g_1 \, x \, g_2^{-1}$$

which defines a left action of G on X. G - invariant Hilbert subspaces $H \hookrightarrow \mathcal{D}'(X)$ are simply called bi- invariant. The reproducing kernels of bi-invariant spaces are of the form

$$K(\varphi \otimes \overline{\psi}) = \langle T, \tilde{\varphi} * \psi \rangle$$

where T is a central positive definite distribution (δ if $H = L^2(X)$). Then since $\tilde{T} = T$ it is easily seen that $\tilde{H} = H$ for every bi-invariant $H \subset \mathcal{D}'(G_0)$. Thus by theorem E $(G_0 \times G_0, G_0)$ is multiplicity free. In the case of type I groups the existence and uniqueness of the decomposition of $L^2(G_0)$ into minimal bi-invariant spaces (property 4. or 5.) gives a new proof of the existence and uniqueness of Plancherel measure (cf. [5]).

b) Let $X = U(p,q;\mathbb{F})/U(1;\mathbb{F}) \times U(p-1,q;\mathbb{F})$, $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, be one of the hyperbolic spaces analysed by J. Faraut in [3]. Then with the methods introduced by Faraut it is easy to show that every G - invariant distribution on $X \times X$ is symmetric. Since the reproducing kernels of G - invariant Hilbert subspaces also have the Hermitian symmetry they are real. Consequently in this case $H = \overline{H}$ for every G - invariant Hilbert subspace of $\mathcal{D}'(X)$, and so by theorem E, (G,X) is multiplicity free.

Let us finally state the theorem which explains the terminology in the title.

Theorem F Let $X = G/_{G_0}$, G_0 being a compact subgroup. Then (G,X) is multiplicity free if and only if (G,G_0) is a Gelfand pair, i.e. the convolution subalgebra of $L^1(G)$ composed of the functions which are left and right G_0 - periodic, is abelian.

Consequently, if $(G,G/_{G_0})$ is multiplicity free we also refer to (G,G_0) as a generalised Gelfand pair.

Proof of Theorem F: We may identify $L^2(X,dx)$ with the space $L^2(G)_{\cdot,G_0}$ of right G_0 - periodic L^2 functions on G. The algebra $L^1_{G_0,G_0}$ of left and right G_0 - periodic function on G acts on $L^2_{\cdot,G_0}$ by convolution from the right, the function being uniquely determined by the corresponding operator. These convolution operators commute with left translations, and it can be shown that actually they generate the von Neumann algebra $(G|L^2)'$. Similar descriptions hold for the other commutants $(G|H)'$ in theorem A. Since the proof of the fact that the commutant is generated by these convolutions operators (with the help of the Segal-Godement commutativity theorem) is too long we resort to the following : If (G,X) is multiplicity free $(G|L^2)'$ is abelian, and so (G,G_0) is a Gelfand pair. On the other hand, if (G,G_0) is a Gelfand pair, it is known that the extreme kernels are the spherical functions of positive type (they form a Souslin section of $\text{ext}(\Gamma_G)$) and it is known that they yield inequivalent irreducible representations. Thus condition 1. of theorem A is satisfied.

Remark If G is connected and G_0 is compact it can be shown that (G,G_0) is a Gelfand pair if and only if the algebra of G - invariant differential operators is commutative [11]. On the other hand A. Lichnerowicz [6] and S. Helgason have shown that on a pseudo-Riemannian symmetric space the algebra of invariant differential operators is commutative. One may wonder therefore if such a pseudo-Riemannian space is multiplicity free. The first proof that this is not the case, known to the author, has been communicated to him by G. van Dijk, who has shown that $(SO_o(1,n), SO_o(1,n-1))$ is not a generalised Gelfand pair for $n>1$ ([1]).

5. Proof of Theorem A

1. ⇒ 2. (proof without measure theoretic details). Let A_1 be maximal commutative algebra in the commutant $(G|H)'$ of the operators $u(g)|_H$, $g\in G$. Let $V\in(G|H)'$ be any unitary operator, and let $A_2 = V A_1 V^{-1}$. Let $P^{(1)}$ and $P^{(2)}$ be the corresponding spectral measures, defined on the Borel sets of the spectrum S of A_1 and A_2. Then $P_A^{(2)} = V P_A^{(1)} V^{-1}$. Let m be a positive measure on S equivalent to $P^{(1)}$ and to $P^{(2)}$. Finally let

$$H = \int^{\oplus} H_s^{(1)} dm(s) = \int^{\oplus} H_s^{(2)} dm(s)$$

be the diagonalisations of A_1 and A_2 (corollary 6). Then by corollary 7 the spaces $H_s^{(i)}$ are G-invariant almost everywhere, and by Mautner's theorem they are almost all irreducible (minimal invariant). Now the operator V decomposes (cf. [2]) :

$$V = \int V_s \, dm(s)$$

where $V_s : H_s^{(1)} \to H_s^{(2)}$ is unitary.

Also it can be shown that V_s intertwines the action of G on $H_s^{(1)}$ and on $H_s^{(2)}$. Thus condition 1. implies that $H_s^{(2)} = \alpha(s) H_s^{(1)}$ for a.a.s, and by Schur's lemma $V_s = \lambda(s) I_s$, with $|\lambda(s)|^2 = \alpha(s)$, for a.a.s. This implies V is a diagonal operator, i.e. $V\in A_1$. Thus $A_1 = (G|H)'$, which proves that the commutant is commutative.

2. ⇒ 3. This implication has been proved in [13].

3. ⇒ 4. This follows from the general theory of integral representation [12][16].

4. ⇒ 5. The existence of the direct integral decomposition follows from proposition 9. Thus the existence and uniqueness follows a fortiori from 4.

5. ⇒ 1. This argument has also been given before [5]. If H_1 and H_2 are minimal invariant their intersection $H = H_1 \cap H_2$, with the norm $\|h\|_H^2 = \|h\|_{H_1}^2 + \|h\|_{H_2}^2$ is invariant and $H \leq H_i$. Therefore $K \leq K_i$, so the K_i being extremal K must be zero. Thus $H_1 \cap H_2 = (0)$ and the sum $H_1 + H_2$ is direct. If the representations of G in H_1 and H_2 are equivalent one easily constructs two other minimal invariant spaces H_1' and H_2' such that $H_1 + H_2 = H_1' + H_2'$. This contradicts the assumption of uniqueness in 5.

6. Closing remark : much of the above is valid in the more general situation where $\mathcal{D}'(X)$ is replaced by a locally convex space, E , together with a representation

G→GL(E). For instance E might be a space of vector valued distributions (cf. [9]) and it might happen that only a subgroup of G is generated by diffeomorphisms of X, as seems to happen in theoretical physics.

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MULTIPLICATION AND CONVOLUTION OPERATORS BETWEEN SPACES OF DISTRIBUTIONS

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Multiplication (convolution) operators between spaces of distributions are continuous linear maps which on the subspace $\mathcal{D}$ act as multiplication (convolution). In section one we fix our notation and recall some basic facts from distribution theory. The historical background of our topic and some connections with different branches of analysis are indicated in section two. In section three we develop to some extent a theory of the spaces $\mathcal{O}_M(E,F)$ and $\mathcal{O}'_C(E,F)$ of multiplication operators and convolution operators from E to F, respectively. These general results are used in section four to work out some examples, whereas some further results are mentioned in section five.

1. NOTATION AND DEFINITIONS

For spaces of functions and distributions, as well as for general locally convex spaces we use the standard notation of HORVÁTH[31] and SCHWARTZ[60]. The symbol Ω always stands for an arbitrary open subset of $\mathbb{R}^n$. For $T\in\mathcal{D}(\Omega)'$ and $\varphi\in\mathcal{E}(\Omega)$ the product $T\cdot\varphi\in\mathcal{D}(\Omega)'$ is defined by

$$\langle T\cdot\varphi,\psi\rangle := \langle T,\varphi\cdot\psi\rangle \quad (\psi\in\mathcal{D}(\Omega)).$$

For $T\in\mathcal{D}(\mathbb{R}^n)'$ and $\varphi\in\mathcal{D}(\mathbb{R}^n)$ the convolution $T*\varphi\in\mathcal{D}(\mathbb{R}^n)'$ is defined by

$$\langle T*\varphi,\psi\rangle := \langle T,\check{\varphi}*\psi\rangle \quad (\psi\in\mathcal{D}(\mathbb{R}^n)),$$

where $\check{\varphi}(x):=\varphi(-x)$ $(x\in\mathbb{R}^n)$, and $\varphi*\psi$ is the usual convolution. We actually have $T*\varphi\in\mathcal{E}(\mathbb{R}^n)$ $(\varphi\in\mathcal{D}(\mathbb{R}^n),\ T\in\mathcal{D}(\mathbb{R}^n)')$, and the map $\{T\}:\mathcal{D}(\mathbb{R}^n)\longrightarrow\mathcal{E}(\mathbb{R}^n)$, $\varphi\longmapsto T*\varphi$, is continuous for all $T\in\mathcal{D}(\mathbb{R}^n)'$. Thus the transpose $\{T\}^t:\mathcal{E}(\mathbb{R}^n)'\longrightarrow\mathcal{D}(\mathbb{R}^n)'$ is defined and continuous. We define $T*S:=\{\check{T}\}^t(S)$ $(T\in\mathcal{D}(\mathbb{R}^n)',$ $S\in\mathcal{E}(\mathbb{R}^n)')$, i.e. $\langle T*S,\varphi\rangle = \langle S,\check{T}*\varphi\rangle = \langle T,\check{S}*\varphi\rangle$, where the reflection on $\mathcal{D}(\mathbb{R}^n)'$ is defined by $\langle\check{T},\varphi\rangle := \langle T,\check{\varphi}\rangle$ $(\varphi\in\mathcal{D}(\mathbb{R}^n))$.

The FOURIER transformation $\mathcal{F}:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n)$ is defined by

$$\mathcal{F}(\varphi)(y) := \int_{\mathbb{R}^n}\varphi(x)e^{-2\pi i x\cdot y}dx \quad (y\in\mathbb{R}^n).$$

We use the same symbol to denote the FOURIER transformation on $\mathcal{S}(\mathbb{R}^n)'$, which is defined by

$$\langle \mathcal{F}(T),\varphi\rangle := \langle T,\mathcal{F}(\varphi)\rangle \quad (T\in\mathcal{S}(\mathbb{R}^n)', \varphi\in\mathcal{S}(\mathbb{R}^n)).$$

Moreover, the restrictions of $\mathcal{F}: \mathcal{S}(\mathbb{R}^n)' \longrightarrow \mathcal{S}(\mathbb{R}^n)'$ to various subspaces of $\mathcal{S}(\mathbb{R}^n)'$ will also be denoted by $\mathcal{F}$. By τ_h we denote the translation operator on $\mathcal{D}(\mathbb{R}^n)$, $\tau_h(\varphi)(x) := \varphi(x-h)$ $(h\in\mathbb{R}^n, \varphi\in\mathcal{D}(\mathbb{R}^n))$, and on $\mathcal{D}(\mathbb{R}^n)'$, $\langle\tau_h(T),\varphi\rangle := \langle T,\tau_{-h}(\varphi)\rangle$ $(T\in\mathcal{D}(\mathbb{R}^n)', h\in\mathbb{R}^n)$.

The basic properties of the above operations as described in SCHWARTZ[60, Ch. V, Ch. VI, Ch. VII] will be used without specific references. We also use a notation introduced by L. SCHWARTZ (e.g. [59, p. 2]): $\exp(2\pi i\hat{x}\cdot y)$ denotes the function $x\longmapsto \exp(2\pi i x\cdot y)$, $\varphi(-\hat{x})$ denotes the function $\check{\varphi}$ ... etc.

By $L^o(\Omega)$ we denote the space of all (equivalence classes of) LEBESGUE measurable functions on $\Omega\subset\mathbb{R}^n$ provided with the (non locally convex) complete metrizable topology of convergence in measure on all subsets of finite measure (BOURBAKI[5, p. 194 ff]). By $C_o(\Omega)$ we denote the space of continuous functions vanishing at infinity. Let E be any subset of $\mathcal{D}'$. We put

$$E_{loc} := \{S\in\mathcal{D}';\ S\cdot\varphi\in E\quad(\varphi\in\mathcal{D})\},$$
$$E_{comp} := \{T\in E;\ \mathrm{supp}(T)\ \text{ is compact}\},$$
$$E_{reg} := \{T*\varphi;\ T\in E,\ \varphi\in\mathcal{D}\},\ \text{and}$$
$$\check{E} := \{\check{T};\ T\in E\}.$$

Let E and F be locally convex spaces. By $L(E,F)$ we denote the space of all continuous linear maps $A:E\longrightarrow F$. If it is necessary to mention the topology $\mathcal{R}$ of the locally convex space E, we write $(E,\mathcal{R})$ or $E_{\mathcal{R}}$. By $L_s(E,F)$ and $L_b(E,F)$ we denote the space $L(E,F)$ provided with the topology of pointwise convergence, and the topology of uniform convergence on all bounded subsets of E, respectively.

We now give the precise definition for multiplication operators and convolution operators between spaces of distributions.

(1.1) Definition (SCHWARTZ[59, p. 69]). Let E and F be locally convex spaces of distributions on Ω, and let E be normal (HORVÁTH [31, p. 319, Def. 3]). A distribution $S\in\mathcal{D}(\Omega)'$ is called a multiplication operator (or multiplier) from E into F if there exists a continuous linear map $[S]:E\longrightarrow F$ such that $[S](\varphi) = S\cdot\varphi$ holds for all $\varphi\in\mathcal{D}(\Omega)\subset E$. By $\mathcal{O}_M(E,F)$ we denote the space of all multiplication operators from E into F.

The definition of a convolution operator is similar.

(1.2) Definition (SCHWARTZ[58, exp. n° 11],[59, p. 72]). Let E and F be locally convex spaces of distributions on $\mathbb{R}^n$, and let E be normal. A distribution $S \in \mathcal{D}(\mathbb{R}^n)'$ is called a convolution operator (or convolutor) from E into F if there exists a continuous linear map $\{S\}: E \longrightarrow F$ such that $\{S\}(\varphi) = S * \varphi$ holds for all $\varphi \in \mathcal{D}(\mathbb{R}^n) \subset E$. By $\mathcal{O}_C'(E,F)$ we denote the space of all convolution operators from E into F.

If E = F we write $\mathcal{O}_M(E)$ and $\mathcal{O}_C'(E)$ instead of $\mathcal{O}_M(E,E)$ and $\mathcal{O}_C'(E,E)$, respectively.

By definition, the spaces $\mathcal{O}_M(E,F)$ and $\mathcal{O}_C'(E,F)$ can be considered as subspaces of $\mathcal{D}'$ and as subspaces of L(E,F). Since the continuous linear maps $[S]: E \longrightarrow F$ and $\{S\}: E \longrightarrow F$ are uniquely determined by the distribution S and vice versa, we usually do not distinguish between the continuous linear maps and their generating distributions.

Before we start the investigation of these spaces in section three, we want to present a few motivations as well as some indications of the historical development of the subject.

2. MOTIVATIONS AND SOME HISTORICAL REMARKS

The theory of multiplication and convolution operators has its origin in the following problem:

(2.1) Let $\mathcal{H}$ be a subspace of $L^1(\mathbb{T})$, where $\mathbb{T}$ denotes the one dimensional torus. What are the conditions a sequence $(c_k; k \in \mathbb{Z})$ in $\mathbb{C}$ must satify such that for all $f \in \mathcal{H}$ the sequence

$$[c](\mathcal{F}(f)) := (c_k \cdot \mathcal{F}(f)(k); k \in \mathbb{Z})$$

is the FOURIER transform of some function $g \in \mathcal{H}$?

Under suitable restrictions, $c = (c_k; k \in \mathbb{Z})$ will be the FOURIER transform of a measure μ on $\mathbb{T}$, and we will have

$$[c](\mathcal{F}(f)) = \mathcal{F}(g) = \mathcal{F}(\mu * f) = [\mathcal{F}(\mu)](\mathcal{F}(f)) \quad (f \in \mathcal{H}),$$

i.e. $\mathcal{O}_M(\mathcal{F}(\mathcal{H}))$ is the space of all those sequences $c = (c_k; k \in \mathbb{Z})$, and $\mathcal{O}_C'(\mathcal{H}) = \mathcal{F}^{-1}(\mathcal{O}_M(\mathcal{F}(\mathcal{H})))$ is the corresponding space of convolution operators.

In the above generality the problem was first treated by FEKETE[18] in 1923. Let $\mathcal{BV}(\mathbb{T})$, $\mathcal{AC}(\mathbb{T})$, and $\mathcal{C}(\mathbb{T})$ denote the space of functions $f: \mathbb{T} \longrightarrow \mathbb{C}$ which are of bounded variation, absolutely continuous, and continuous, respectively. One of the main results of FEKETE is $\mathcal{O}_C'(L^1(\mathbb{T})) = \mathcal{O}_C'(L^\infty(\mathbb{T})) = \mathcal{O}_C'(\mathcal{BV}(\mathbb{T})) = \mathcal{O}_C'(\mathcal{AC}(\mathbb{T})) = \mathcal{O}_C'(\mathcal{C}(\mathbb{T}))$.

Still earlier, results on individual multipliers were obtained by

YOUNG[79, 80] in 1913, MAZURKIEWICZ[49], STEINHAUS[67, p. 217 ff] in 1915, and problem (2.1) was also investigated by STEINHAUS[68, p. 210], SZIDON[70], BOCHNER[4], KACZMARZ,MARCINKIEWICZ[36], and MARCINKIEWICZ [44]. We refer to HEWITT, ROSS[27, sections 35, 36 and p. 412], EDWARDS[16, Ch. 16], ZYGMUND[82, Ch. IV, § 11 and p. 378], LARSEN[41], and STEIN, WEISS[66, p. 257 ff] for further results and references.

We note the following aspect of the above problem (2.1): The spaces $\mathcal{F}(L^p)$ are still not too well understood. Thus every special multiplier $f \in \mathcal{O}_M(\mathcal{F}(L^p))$ extends our knowledge on the richness of $\mathcal{F}(L^p)$.

Already the result of FEKETE shows some special features of the spaces $\mathcal{O}_M(E,F)$ and $\mathcal{O}'_C(E,F)$.

- The spaces $\mathcal{O}_M(E,F)$ and $\mathcal{O}'_C(E,F)$ are not too sensitive with respect to a change in the arguments E and F.
- Dual spaces often have the same spaces of multiplication (convolution) operators.
- If the FOURIER transformation is defined on E and on F, it provides a linear isomorphism between $\mathcal{O}'_C(E,F)$ and $\mathcal{O}_M(\mathcal{F}(E),\mathcal{F}(F))$.

<u>(2.2)</u> Summation problems. Suppose we are given the FOURIER transform $\mathcal{F}(f)$ of a function $f \in L^p(\mathbb{R}^n)$ with $p \in [1,\infty)$. To recover f from $\mathcal{F}(f)$ we could proceed as follows: Let $m \in L^\infty_{comp}(\mathbb{R}^n)$ be continuous at $0 \in \mathbb{R}^n$, and suppose $m(0)=1$. For $\rho > 0$ we put $M_\rho(m)(x) := m(\rho \cdot x)$ and we define an operator T_ρ in (say) $L^2(\mathbb{R}^n)$ by

$$\mathcal{F}(T_\rho(f)) := M_\rho(m) \cdot \mathcal{F}(f) \quad (f \in L^2 \cap L^p,\ \rho > 0),$$

thus $T_\rho = \mathcal{F}^{-1} \circ [M_\rho(m)] \circ \mathcal{F} = \{\mathcal{F}^{-1} \circ M_\rho(m)\}$. Using $\mathcal{F}^{-1} \circ M_\rho = \rho^{-n} \cdot M_\rho \circ \mathcal{F}^{-1}$ we obtain

$$\{\mathcal{F}^{-1} \circ M_\rho(m)\} = M_{1/\rho} \circ \{\mathcal{F}^{-1}(m)\} \circ M_\rho = (M_\rho)^{-1} \circ \{\mathcal{F}^{-1}(m)\} \circ M_\rho.$$

Observing that $\rho^{n/p} \cdot M_\rho$ is an isometry in $L^p(\mathbb{R}^n)$ $(\rho > 0)$, we therefore obtain that T_ρ is the conjugation of $\{\mathcal{F}^{-1}(m)\}$ by the L^p-isometry $\tilde{M}_\rho := \rho^{n/p} \cdot M_\rho$ $(\rho > 0)$. Thus, if $T_1 = \{\mathcal{F}^{-1}(m)\}$ has an extension to a continuous linear operator $\tilde{T}_1 : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$, we obtain $\tilde{T}_\rho(f) \longrightarrow f$ $(\rho \longrightarrow 0)$ (cf. STEIN[65, p. 94 and p. 99]).

Via the POISSON summation formula results of the above type imply results on the summability of multiple FOURIER series, such as

$$f_j := \sum_{\|k\|_\infty \leq j} \mathcal{F}(f)(k) \cdot \exp(2\pi i k \cdot \hat{\theta}) \longrightarrow f \quad (j \longrightarrow \infty)$$

in $L^p(\mathbb{T}^n)$ $(p \in (1,\infty))$. Here m is the characteristic function of the $\|\cdot\|_\infty$-unit ball in $\mathbb{R}^n$ (cf. STEIN,WEISS[66, p. 260 ff], STEIN[65, p. 99]). FEFFERMAN[17] proved that if m equals the characteristic function of the $\|\cdot\|_2$-unit ball in $\mathbb{R}^n$, the operator $\{\mathcal{F}^{-1}(m)\}$ is unbounded on $L^p(\mathbb{R}^n)$ for $n>1$ and $p\neq 2$, whereas it was known for a long time that the corresponding operator for the $\|\cdot\|_\infty$-unit ball in $\mathbb{R}^n$ is bounded in $L^p(\mathbb{R}^n)$ for all $p\in(1,\infty)$ (STEIN[65, p. 99, 4.1], SCHWARTZ[57]).

The spaces $\mathcal{O}_C'(L^r,L^s)$ were investigated in detail by HÖRMANDER [30]. Nevertheless a complete description of these spaces seems still to be unknown.

(2.3) Let $P(\partial)$ be a homogeneous elliptic partial differential operator of order k and let $\alpha\in\mathbb{N}_o^n$ satisfy $|\alpha| = k$. The fact that $\hat{x}^\alpha/P(\hat{x})$ is a multiplier on $\mathcal{F}(L^p)$ for all $p\in(1,\infty)$ yields the a priori estimate

$$\|\partial^\alpha f\|_p \leq A_p\|P(\partial)(f)\|_p \quad (f\in\mathcal{D}^k(\mathbb{R}^n))$$

(cf. STEIN[65, p. 77]). The existence of bounded rational functions which are multipliers on no $\mathcal{F}(L^p)$ ($p\in[1,\infty)$) was proved by LITTMAN, McCARTHY, RIVIÈRE[42]. A continuation of this path ultimately leads to the theory of pseudo differential operators (TAYLOR[71]).

We now mention shortly several other applications.

(2.4) The KÖTHE function space Λ defined by a subspace E of $L^1_{loc}(\Omega)$ (DIEUDONNÉ[14]) is just $\mathcal{O}_M(E,L^1(\Omega))$. In particular, the space $S_{ar}(\mathbb{R}^n)$ of absolutely regular tempered distributions on $\mathbb{R}^n$ coincides with $\mathcal{O}_M(\mathcal{S},L^1)$ (cf. [9, 10]).

(2.5) The transformation of coordinates in (say) the SOBOLEV spaces $W^{m,p}$ leads inevitably to the multiplier problem for these spaces (ADAMS[1, p. 63], PALAIS[52, p. 31], WLOKA[76, p. 80]). Similarly, the study of so called superposition operators (or NEMITSKY operators) on SOBOLEV spaces leads to the investigation of multiplication operators on these spaces (MARCUS, MIZEL[45], PALAIS[52, p. 31]).

(2.6) Let A be a closed densely defined operator in $L^2(\mathbb{R}^n)$ which commutes with all translations. Then A is (normal and) unitarily equivalent via FOURIER transform to a multiplication operator (cf. the lecture of E. THOMAS in this volume; a proof was given independently by J. VOIGT, unpublished). More general, by the spectral representation theorem, the normal operators on a HILBERT space are exactly the operators which are unitarily equivalent to a multiplication operator on some L^2.

(2.7) The definition of convolvability for two distributions S, T in $\mathcal{D}(\mathbb{R}^n)'$ given by CHEVALLEY[6, p. 112] in 1950 connects regularization, multiplication, and convolution:
$S,T\in\mathcal{D}(\mathbb{R}^n)'$ are convolvable: $\Longleftrightarrow$ $(S*\varphi)\cdot(\check{T}*\psi)\in L^1(\mathbb{R}^n)$ for all

$\varphi,\psi\in\mathcal{D}(\mathbb{R}^n)$.

(2.8) In problems of mathematical physics it is often important to know on which spaces potentials or fundamental solutions of partial differential operators act continuously via convolution (HORVÁTH[33, 34], ORTNER[50,51], MALGRANGE[43, p.288, Thm. 1], WLADIMIROV[75, §22]).

We finally mention that characterizations of distribution spaces by their regularizations, e.g. $T\in\mathcal{D}'_{L^p}(\mathbb{R}^n)\Longleftrightarrow T*\varphi\in L^p(\mathbb{R}^n)$ for all $\varphi\in\mathcal{D}(\mathbb{R}^n)$ (SCHWARTZ[60, p.201]), and the fundamental approximation procedure by "cutting and regularization" (SCHWARTZ[59, p.7/8]) also fall into the scope of our subject.

3. GENERAL RESULTS ON THE SPACES $\mathcal{O}_M(E,F)$ and $\mathcal{O}'_C(E,F)$

The following observations are obvious but nevertheless very useful.

(3.1) (a) If E,F,G, and H are spaces of distributions, E,G normal, the continuous inclusions

$$G\hookrightarrow E \quad\text{and}\quad F\hookrightarrow H$$

imply

$$\mathcal{O}_M(E,F)\ \begin{matrix}\subset \mathcal{O}_M(G,F) \subset\\ \subset \mathcal{O}_M(E,H) \subset\end{matrix}\ \mathcal{O}_M(G,H),$$

and the corresponding inclusions for $\mathcal{O}'_C$.

(b) For every subspace $H\subset\mathcal{O}_M(E,F)$ we have the bilinear map $H\times E\longrightarrow F$, $(S,T)\longmapsto[S](T)$. It often happens that this bilinear map is separately continuous. From $[\varphi](T)=T\cdot\varphi$ $(T\in E,\varphi\in\mathcal{D}\subset H)$ we then obtain $E\subset\mathcal{O}_M(H,F)$.

Another fundamental tool is the following:

(3.2) Suppose we have a continuous injection $H\hookrightarrow F$ and we know $[S](E)\subset H$ for all $S\in\mathcal{O}_M(E,F)$ (or $\{S\}(E)\subset H$ for all $S\in\mathcal{O}'_C(E,F)$). In many cases a closed graph theorem is available for the pair (E,H) which then yields the continuity of $[S]:E\longrightarrow H$ for all $S\in\mathcal{O}_M(E,F)$ and thus $\mathcal{O}_M(E,F)=\mathcal{O}_M(E,H)$ (or $\mathcal{O}'_C(E,F)=\mathcal{O}'_C(E,H)$). More general, a space of distributions H is said to have property (C) if for every barrelled space E and every continuous linear map $A:E\longrightarrow\mathcal{D}'$ the inclusion $A(E)\subset H$ implies the continuity of $A:E\longrightarrow H$ (cf. SHIRAISHI[62, p. 22]). In loc. cit. it is shown that many spaces of distributions have property (C). In particular, H has property (C) if $\mathcal{D}$ is strictly dense in $(H',\sigma(H',H))$ (SHIRAISHI[63, p. 176]).

(3.3) For $S\in\mathcal{O}_M(E,F)$ the map $[S]^t:(F',\tau(F',F))\longrightarrow(E',\tau(E',E))$

is continuous. If F is normal, then $(F', \tau(F',F))$ is normal too, and we obtain

$_{E'}\langle[S]^t(\varphi),\psi\rangle_E = {}_{F'}\langle\varphi,[S](\psi)\rangle_F = {}_{\mathcal{D}}\langle\varphi, S\cdot\psi\rangle_{\mathcal{D}'} = {}_{\mathcal{D}'}\langle S, \varphi\cdot\psi\rangle_{\mathcal{D}} =$
${}_{\mathcal{D}'}\langle S\cdot\varphi,\psi\rangle_{\mathcal{D}}$ for all $\varphi,\psi\in\mathcal{D}(\Omega)$.

Thus $[S]^t(\varphi) = S\cdot\varphi\in E'$ for all $\varphi\in\mathcal{D}(\Omega)$. This implies $\mathcal{O}_M(E,F)\subset\mathcal{O}_M(F'_\tau,E'_\tau)$ and $\mathcal{O}_M(E,F) = \mathcal{O}_M(F'_\tau,E'_\tau)$ if E is a MACKEY space, i.e. $(E,\mathcal{R}) = (E,\tau(E,E'))$. Moreover we obtain

$$\mathcal{O}_M(E,F)\subset(F)_{loc}\cap(E')_{loc}$$

if E and F are normal.

In the calculations, the above inclusion yields local regularity results for the elements of $\mathcal{O}_M(E,F)$, e.g.: $\mathcal{O}_M(\mathcal{D}',\mathcal{D}') = \mathcal{O}_M(\mathcal{D},\mathcal{D})\subset$ $\subset(\mathcal{D}')_{loc}\cap(\mathcal{D})_{loc} = \mathcal{E}$. By SCHWARTZ[60,p. 119] we thus obtain $\mathcal{O}_M(\mathcal{D}',\mathcal{D}') = \mathcal{O}_M(\mathcal{D},\mathcal{D}) = \mathcal{E}(\Omega) = \mathcal{O}_M(\mathcal{E},\mathcal{E})$, and

$$\mathcal{E}(\Omega)\subset\mathcal{O}_M(E,\mathcal{D}') \text{ for all normal spaces } E.$$

Another example is the following: $\mathcal{O}_M(L^1_{comp},\mathcal{D}')\subset(\mathcal{D}')_{loc}\cap(L^\infty_{loc})_{loc} =$ $= L^\infty_{loc}\Rightarrow\mathcal{O}_M(L^1_{comp},\mathcal{D}') = \mathcal{O}_M(L^1,\mathcal{D}') = \mathcal{O}_M(L^1,L^1_{loc}) = \mathcal{O}_M(L^1_{loc},\mathcal{D}') =$ $= \mathcal{O}_M(L^1_{loc},L^1_{loc}) = L^\infty_{loc}$.

We note the corresponding results for convolution operators.

<u>(3.4)</u> Let $S\in\mathcal{O}'_C(E,F)$ and let F be a normal space.

$_{E'}\langle\{S\}^t(\varphi),\psi\rangle_E = {}_{F'}\langle\varphi,\{S\}(\psi)\rangle_F = {}_{\mathcal{D}}\langle\varphi,S*\psi\rangle_{\mathcal{D}'} = {}_{\mathcal{D}'}\langle S,\check{\psi}*\varphi\rangle_{\mathcal{D}} =$
$={}_{\mathcal{D}'}\langle\check{S}*\varphi,\psi\rangle_{\mathcal{D}}$ for all $\varphi,\psi\in\mathcal{D}(\mathbb{R}^n)$.

Thus $\{S\}^t(\varphi) = \check{S}*\varphi\in E'$ for all $\varphi\in\mathcal{D}(\mathbb{R}^n)$. This implies $\mathcal{O}'_C(E,F)^\vee\subset\mathcal{O}'_C(F'_\tau,E'_\tau)$ and $\mathcal{O}'_C(E,F)^\vee = \mathcal{O}'_C(F'_\tau,E'_\tau)$ if E is a MACKEY space. Moreover we obtain

$$\mathcal{O}'_C(E,F)_{reg}\subset F\cap(E')^\vee, \text{ i.e. } S*\varphi\in F\cap(E')^\vee$$
$$\text{for all } S\in\mathcal{O}'_C(E,F),\ \varphi\in\mathcal{D}(\mathbb{R}^n).$$

In the calculations the above inclusion yields growth conditions for the elements of $\mathcal{O}'_C(E,F)$, e.g.: $\mathcal{O}'_C(\mathcal{E},\mathcal{D}')_{reg}\subset\mathcal{D}'\cap(\mathcal{E}')^\vee = \mathcal{E}'$ $\Rightarrow\mathcal{O}'_C(\mathcal{E},\mathcal{D}')\subset\mathcal{E}(\mathbb{R}^n)'\Rightarrow\mathcal{O}'_C(\mathcal{E},\mathcal{D}') = \mathcal{O}'_C(\mathcal{D},\mathcal{D}) = \mathcal{O}'_C(\mathcal{D}',\mathcal{D}') =$ $= \mathcal{E}(\mathbb{R}^n)'$, and

$$\mathcal{E}(\mathbb{R}^n)'\subset\mathcal{O}'_C(E,\mathcal{D}') \text{ for all normal spaces } E.$$

By (3.1) we thus obtain $\mathcal{E}(\mathbb{R}^n)' = \mathcal{O}'_C(\mathcal{D}',\mathcal{D}')\subset\mathcal{O}'_C(L^1_{loc},\mathcal{D}')\subset$ $\subset\mathcal{O}'_C(\mathcal{E}^o,\mathcal{D}')\subset\mathcal{O}'_C(\mathcal{E},\mathcal{D}')\subset\mathcal{E}(\mathbb{R}^n)' = \mathcal{O}'_C(\mathcal{E},\mathcal{E})$.

The inclusions (3.4) can also be used to prove a representation theorem for $\mathcal{O}_C'(E,F)$ by the so called parametrix method.

(3.5) Example $\mathcal{O}_C'(\mathcal{S})$. Let $\mathcal{S}_m^o(\mathbb{R}^n)$ denote the BANACH space of all continuous functions f such that $(1+|\hat{x}|^2)^m f \in C_o(\mathbb{R}^n)$. By K we denote the closed unit ball in $\mathbb{R}^n$. Now let $T \in \mathcal{O}_C'(\mathcal{S}') = \mathcal{O}_C'(\mathcal{S})$ and let $m \in \mathbb{N}_o$ be given. Then $\{T\}: \mathcal{D}_K(\mathbb{R}^n) \longrightarrow \mathcal{S}_m^o(\mathbb{R}^n)$ is continuous for all $m \in \mathbb{N}_o$. Therefore, for every $m \in \mathbb{N}_o$ there exists $p = p(m) \in \mathbb{N}_o$ such that $\{T\}$ has a continuous extension $\{T\}: \mathcal{D}_K^p \longrightarrow \mathcal{S}_m^o$. According to SCHWARTZ[60, p. 47 and p. 191] there exist $k=k(m,p)$, $\xi \in \mathcal{D}_K^p(\mathbb{R}^n)$, and $\zeta \in \mathcal{D}(\mathbb{R}^n)$ such that $\delta = \Delta^k \xi - \zeta$. We thus obtain

$$T = T * \delta = T * (\Delta^k \xi) - T * \zeta = \sum_{|\alpha| \leq 2k} c_\alpha \partial^\alpha (T * \xi) - T * \zeta ,$$

i.e.: $\forall\, m \in \mathbb{N}_o \ \exists\, k \in \mathbb{N}_o$, $(f_\alpha;\, |\alpha| \leq 2k)$ in $\mathcal{S}_m^o(\mathbb{R}^n)$: $T = \sum_{|\alpha| \leq 2k} \partial^\alpha f_\alpha$.

Using the inequality $(1+|x-y|^2)^{-m} \leq 2^m (1+|x|^2)^{-m} (1+|y|^2)^m$ $(x,y \in \mathbb{R}^n, m \in \mathbb{N}_o)$ it is easy to verify that the above necessary condition implies $T * \varphi \in \mathcal{S}(\mathbb{R}^n)$ $(\varphi \in \mathcal{D}(\mathbb{R}^n))$ and that $\{T\}: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is continuous with respect to the topology induced by $\mathcal{S}(\mathbb{R}^n)$ on $\mathcal{D}(\mathbb{R}^n)$.

(3.6) If there exists a continuous inclusion $F \hookrightarrow \mathcal{E}^o(\mathbb{R}^n)$, the relation $\langle \check{T}, \varphi \rangle = (\check{T} * \check{\varphi})(0) = (T * \varphi)(0) = \langle \delta, T * \varphi \rangle$ $(\varphi \in \mathcal{D}(\mathbb{R}^n)$, $T \in \mathcal{D}(\mathbb{R}^n)'$; cf. SCHWARTZ[60, p. 167/168]) shows $\mathcal{O}_C'(E,F) \subset (E')^\vee$.

We give a simple application of this inclusion which is due to MALGRANGE[43, p. 289/290]. It is easy to prove $\mathcal{O}_C'(L^2_{comp}, L^2_{loc}) \subset \mathcal{O}_C'(W^{m,2}{}_{comp}, W^{m,2}{}_{loc})$ for all $m \in \mathbb{N}_o$. By SOBOLEV's theorem we have $W^{m,2}{}_{loc}(\mathbb{R}^n) \hookrightarrow \mathcal{E}^o(\mathbb{R}^n)$ for $2m > n$. We thus obtain $\mathcal{O}_C'(L^2_{comp}, L^2_{loc}) \subset W^{-m,2}{}_{loc}(\mathbb{R}^n)$ for $2m > n$.

(3.7) Remark. At the end of the lecture in Paderborn, E. THOMAS asked for a description of $\mathcal{O}_C'(L^2_{comp}, L^2_{loc})$. In loc. cit. MALGRANGE notes that the above inclusion is strict. By (3.2) we obtain $\mathcal{O}_C'(L^2_{comp}, L^2_{loc}) \subset \mathcal{O}_C'(\mathcal{K}, \mathcal{M})$; the inclusion $\mathcal{O}_C'(\mathcal{K}, \mathcal{M}) \subset \mathcal{O}_C'(L^2_{comp}, L^2_{loc})$ was proved by GAUDRY[23, p. 474, Thm.5], and independently by E. THOMAS (unpublished).

The above results explain to some extent why the spaces $\mathcal{O}_M(E,F)$ and $\mathcal{O}_C'(E,F)$ are not too sensitive with respect to changes in the "arguments" E and F.

Characterization theorems.

Right from the definition we obtain

$$\mathcal{O}_M(E,F) = L(E,F) \cap \mathcal{O}_M(\mathcal{D},\mathcal{D}')$$

and

$$\mathcal{O}_C'(E,F) = L(E,F) \cap \mathcal{O}_C'(\mathcal{D},\mathcal{D}').$$

It is thus sufficient to have characterizations for $\mathcal{O}_M(\mathcal{D},\mathcal{D}')$ and $\mathcal{O}_C'(\mathcal{D},\mathcal{D}')$ as subspaces of $L(\mathcal{D},\mathcal{D}')$.

(3.8) Theorem ([9,p. 48]). For a linear map $A:\mathcal{D}(\Omega) \longrightarrow \mathcal{D}(\Omega)'$ the following are equivalent:

(a) $A \circ [\varphi] = [\varphi] \circ A$ for all $\varphi \in \mathcal{D}(\Omega)$.

(b) There exists a unique $S \in \mathcal{D}(\Omega)'$ such that $A(\varphi) = S \cdot \varphi$ for all $\varphi \in \mathcal{D}(\Omega)$.

Proof. (a) $\Longrightarrow$ (b): We first show that A is a local operator. Let $\psi \in \mathcal{D}(\Omega)$ and let U be an arbitrary open neighbourhood of $\mathrm{supp}(\psi)$. We choose $\varphi \in \mathcal{D}(\Omega)$ such that $\mathrm{supp}(\varphi) \subset U$ and $\varphi(x) = 1$ for all $x \in \mathrm{supp}(\psi)$. By (a) we have $A(\psi) = A(\varphi \cdot \psi) = \varphi \cdot A(\psi)$ which implies $\mathrm{supp}(A(\psi)) \subset \mathrm{supp}(\varphi) \subset U$. This shows $\mathrm{supp}(A(\psi)) \subset \mathrm{supp}(\psi)$ for all $\psi \in \mathcal{D}(\Omega)$. Now let $(\psi_j; j \in \mathbb{N})$ be a partition of unity in $\mathcal{D}(\Omega)$ such that $(\mathrm{supp}(\psi_j); j \in \mathbb{N})$ is locally finite. Since A is a local operator, the sequence $(\sum_{j \leq k} A(\psi_j); k \in \mathbb{N})$ is convergent in $(\mathcal{D}(\Omega)', \beta(\mathcal{D}',\mathcal{D}))$. We define $S := \sum_{j \in \mathbb{N}} A(\psi_j)$. For every $\varphi \in \mathcal{D}(\Omega)$ there is a finite subset J of $\mathbb{N}$ such that $\varphi \cdot \psi_j = 0$ for all $j \notin J$. We thus obtain

$$A(\varphi) = A(\sum_{j \in J} \varphi \cdot \psi_j) = \varphi \cdot (\sum_{j \in J} A(\psi_j)) = \varphi(\sum_{j \in \mathbb{N}} A(\psi_j)) = S \cdot \varphi \quad (\varphi \in \mathcal{D}(\Omega)).$$

Now let $S_1, S_2 \in \mathcal{D}(\Omega)'$ satisfy $S_1 \cdot \psi = S_2 \cdot \psi$ for all $\psi \in \mathcal{D}(\Omega)$. Then $S_1 = \sum_{j \in \mathbb{N}} S_1 \cdot \psi_j = \sum_{j \in \mathbb{N}} S_2 \cdot \psi_j = S_2$.

(b) $\Longrightarrow$ (a) is obvious.

The corresponding theorem for $\mathcal{O}_C'$ is well known (SCHWARTZ[60, p. 197/198]).

(3.9) Theorem. For a continuous linear map $A: \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}(\mathbb{R}^n)'$ the following are equivalent:

(a) $\tau_h \circ A = A \circ \tau_h$ for all $h \in \mathbb{R}^n$.

(b) $\partial^\alpha \circ A = A \circ \partial^\alpha$ for all $\alpha \in \mathbb{N}_o^n$.

(c) $A \circ \{\varphi\} = \{\varphi\} \circ A$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(d) There exists a unique $S \in \mathcal{D}(\mathbb{R}^n)'$ such that $A(\varphi) = S * \varphi$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

We note the following consequences.

(3.10) Corollary. (a) $\mathcal{O}_M(E,F)$ and $\mathcal{O}'_C(E,F)$ are closed subspaces of $L_s(E,F)$.
(b) Let H be a normal space of distributions and suppose $\mathcal{D}(\Omega) \subset \subset \mathcal{O}_M(E) \cap \mathcal{O}_M(F)$. Then $\mathcal{O}_M(E,F) \circ \mathcal{O}_M(H,E) \subset \mathcal{O}_M(H,F)$. In particular, $\mathcal{D}(\Omega) \subset \mathcal{O}_M(E)$ implies that $\mathcal{O}_M(E)$ is a subalgebra of $L(E)$.
(c) Let H be a normal space of distributions and suppose $\mathcal{D}(\mathbb{R}^n) \subset \subset \mathcal{O}'_C(E) \cap \mathcal{O}'_C(F)$. Then $\mathcal{O}'_C(E,F) \circ \mathcal{O}'_C(H,E) \subset \mathcal{O}'_C(H,F)$. In particular, $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{O}'_C(E)$ implies that $\mathcal{O}'_C(E)$ is a subalgebra of $L(E)$.

Proof. (a) Let $A:E \longrightarrow F$ be linear and continuous and let $(S_\iota ; \iota \in I)$ be a net in $\mathcal{O}_M(E,F)$ such that $\lim_{\iota\in I}[S_\iota](T) = A(T)$ $(T \in E)$ holds with respect to the topology of F. For all $\varphi, \psi, \eta \in \mathcal{D}(\Omega)$ we then obtain $\langle A(\varphi\cdot\psi) - \varphi\cdot A(\psi), \eta\rangle = \langle \lim_{\iota\in I}[S_\iota](\varphi\cdot\psi) - \varphi\cdot \lim_{\iota\in I}[S_\iota](\psi), \eta\rangle$
$= \lim_{\iota\in I}\langle S_\iota, \varphi\cdot\psi\cdot\eta\rangle - \lim_{\iota\in I}\langle S_\iota, \varphi\cdot\psi\cdot\eta\rangle = 0 \Longrightarrow (A|\mathcal{D}(\Omega)) \in \mathcal{O}_M(\mathcal{D},\mathcal{D}')$
according to (3.8), and thus $A \in \mathcal{O}_M(E,F)$.
The proof for $\mathcal{O}'_C(E,F)$ is similar.

(b) Let $A \in \mathcal{O}_M(E,F)$ and $B \in \mathcal{O}_M(H,E)$. Then $A\circ B:H \longrightarrow F$ is linear and continuous. The inclusion $\mathcal{D}(\Omega) \subset \mathcal{O}_M(E) \cap \mathcal{O}_M(F)$ implies the continuity of the maps $A\circ[\varphi]$, $[\varphi]\circ A:E \longrightarrow F$ $(\varphi \in \mathcal{D}(\Omega))$. Thus $A([\varphi](R)) = [\varphi](A(R))$ holds for all $R \in E$, $\varphi \in \mathcal{D}(\Omega)$. Now let $\varphi, \psi \in \mathcal{D}(\Omega)$. We obtain $(A\circ B)\circ[\varphi](\psi) = A((B\circ[\varphi])(\psi)) = A(([\varphi]\circ B)(\psi)) = A([\varphi](B(\psi))) = [\varphi](A(B(\psi))) = [\varphi]\circ(A\circ B)(\psi)$. Therefore $A\circ B|\mathcal{D}(\Omega) \in \mathcal{O}_M(\mathcal{D},\mathcal{D}')$, and we obtain $A\circ B \in \mathcal{O}_M(H,F)$.

The proof of (c) is similar.

Invariance under differentiation and multiplication.

(3.11) Suppose $\partial^\alpha \in L(E)$ and $\partial^\alpha \in L(F)$ for all $\alpha \in \mathbb{N}_o^n$. If $S \in \mathcal{O}_M(E,F)$, the relation $[\partial_j S] = \partial_j\circ[S] - [S]\circ\partial_j$ $(j=1,\ldots,n)$ on $\mathcal{D}(\Omega)$ shows that $[\partial_j S]: \mathcal{D}(\Omega) \longrightarrow F$ has a continuous extension $[\partial_j S]:E \longrightarrow F$ $(j=1,\ldots,n)$. We thus obtain $\partial^\alpha \mathcal{O}_M(E,F) := \{\partial^\alpha S;\ S \in \in \mathcal{O}_M(E,F)\} \subset \mathcal{O}_M(E,F)$ for all $\alpha \in \mathbb{N}_o^n$.

This elementary observation already enables us to compute $\mathcal{O}_M(\mathcal{S}')$.

(3.12) Example $\mathcal{O}_M(\mathcal{S}')$. By (3.3) we know $\mathcal{O}_M(\mathcal{S}') = \mathcal{O}_M(\mathcal{S}) \subset \subset \mathcal{E}(\mathbb{R}^n)$. A routine approximation argument shows $[\eta](\varphi) = \eta\cdot\varphi$ $(\eta \in \mathcal{O}_M(\mathcal{S}),\ \varphi \in \mathcal{S})$. Let $\eta \in \mathcal{O}_M(\mathcal{S})$, denote $\omega_k := (1+|\hat{x}|^2)^k$ $(k \in \mathbb{Z},\ x \in \mathbb{R}^n)$, and let B be a bounded subset of $\bigcap_{k\in\mathbb{N}_o} L^\infty(\mathbb{R}^n, \omega_k)$. Then there exists $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $|f| \leq \psi$ for all $f \in B$ (cf. CHEVALLEY[6, p. 127, Lemma (3.6)]), which implies $|\eta\cdot f| \leq |\eta\cdot\psi|$

for all $f \in B$. Since $\bigcap_{k\in\mathbb{N}_0} L^\infty(\mathbb{R}^n, \omega_k)$ is metrizable, we obtain the continuity of $[\eta] : \bigcap_{k\in\mathbb{N}_0} L^\infty(\mathbb{R}^n, \omega_k) \longrightarrow L^\infty(\mathbb{R}^n)$. Thus there exists $m \in \mathbb{N}_0$ and $C > 0$ such that $\|\eta \cdot f\|_\infty \leq C\|\omega_m f\|_\infty$ holds for all $f \in \bigcap_{k\in\mathbb{N}_0} L^\infty(\mathbb{R}^n, \omega_k)$. Using a suitable approximate unit $(\theta_j; j\in\mathbb{N})$ in $\mathcal{D}(\mathbb{R}^n)$ we obtain $\theta_j \cdot \omega_{-(m+1)} \to \omega_{-(m+1)}$ $(j \to \infty)$ in $L^\infty(\mathbb{R}^n, \omega_m)$. This shows $\eta \cdot \omega_{-(m+1)} \in L^\infty(\mathbb{R}^n)$. By (3.11) we thus obtain:

$$\forall \eta \in \mathcal{O}_M(\mathcal{S}') = \mathcal{O}_M(\mathcal{S})\ \forall\ \alpha \in \mathbb{N}_0^n\ \exists\ m(\alpha) \in \mathbb{N} : \omega_{-m(\alpha)} \cdot \partial^\alpha \eta \in L^\infty(\mathbb{R}^n).$$

It is nearly obvious that this necessary condition is also sufficient.

<u>(3.13)</u> Let $\varphi \in \mathcal{E}(\Omega)$ satisfy $[\varphi] \in L(E)$ or $[\varphi] \in L(F)$. If $S \in \mathcal{O}_M(E,F)$, the relation $[\varphi \cdot S] = [\varphi] \circ [S] = [S] \circ [\varphi]$ on $\mathcal{D}(\Omega)$ shows that $[\varphi \cdot S] : \mathcal{D}(\Omega) \longrightarrow F$ has a continuous extension $[\varphi \cdot S] : E \longrightarrow F$. We thus obtain $\varphi \cdot \mathcal{O}_M(E,F) \subset \mathcal{O}_M(E,F)$.

For $S \in \mathcal{S}(\mathbb{R}^n)'$ we have $\mathcal{F}(\partial^\alpha S) = (2\pi i \hat{x})^\alpha \cdot \mathcal{F}(S)$. Taking into account the intimate relation between $\mathcal{O}_C'$ and $\mathcal{O}_M$ via FOURIER transform, we might expect a result similar to (3.11) for $\mathcal{O}_C'$ and the multiplication by polynomials.

<u>(3.14)</u> Suppose $[\hat{x}^\alpha] \in L(E)$ and $[\hat{x}^\alpha] \in L(F)$ for all $\alpha \in \mathbb{N}_0^n$. If $S \in \mathcal{O}_C'(E,F)$ the relation $\{\hat{x}_j S\} = [\hat{x}_j] \circ \{S\} - \{S\} \circ [\hat{x}_j]$ $(j=1,\ldots,n)$ on $\mathcal{D}(\mathbb{R}^n)$ (cf. SCHWARTZ[60, p. 169, (VI,4;15)]) shows that $\{\hat{x}_j S\} : \mathcal{D}(\mathbb{R}^n) \longrightarrow F$ has a continuous extension $\{\hat{x}_j S\} : E \longrightarrow F$ $(j=1,\ldots,n)$. We thus obtain $\hat{x}^\alpha \mathcal{O}_C'(E,F) \subset \mathcal{O}_C'(E,F)$ $(\alpha \in \mathbb{N}_0^n)$.

<u>(3.15)</u> Let $\partial^\alpha \in L(E)$ or $\partial^\alpha \in L(F)$ $(\alpha \in \mathbb{N}_0^n)$ be satisfied. If $S \in \mathcal{O}_C'(E,F)$ the relation $\{\partial^\alpha S\} = \partial^\alpha \circ \{S\} = \{S\} \circ \partial^\alpha$ on $\mathcal{D}(\mathbb{R}^n)$ shows that $\{\partial^\alpha S\} : \mathcal{D}(\mathbb{R}^n) \longrightarrow F$ has a continuous extension $\{\partial^\alpha S\} : E \longrightarrow F$. We thus obtain $\partial^\alpha \mathcal{O}_C'(E,F) \subset \mathcal{O}_C'(E,F)$ $(\alpha \in \mathbb{N}_0^n)$.

<u>(3.16) Inclusions for the supports.</u> In general we don't know how $[S]$ or $\{S\}$ acts on elements T of E. If we suppose that to every $T \in E$ and to every uniform neighbourhood U of $\mathrm{supp}(T)$ there exists a sequence $(\varphi_j; j\in\mathbb{N})$ in $\mathcal{D}$ such that $\mathrm{supp}(\varphi_j) \subset U$ $(j \in \mathbb{N})$ and $\varphi_j \to T$ $(j \to \infty)$ in E (e.g. if E has the approximation property with respect to cutting and regularization in the sense of SCHWARTZ [59, p.7/8]), it is easy to prove the inclusions one would expect, namely

$$\mathrm{supp}([S](T)) \subset \mathrm{supp}(S) \cap \mathrm{supp}(T) \quad (S \in \mathcal{O}_M(E,F),\ T \in E)$$

and

$$\mathrm{supp}(\{S\}(T)) \subset \overline{\mathrm{supp}(S) + \mathrm{supp}(T)} \quad (S \in \mathcal{O}_C'(E,F),\ T \in E).$$

Pointwise action of the operators [S] and {S}.

As we already noted above, in general the action of the operators [S] and {S} on elements $T \in E \setminus \mathcal{D}$ can only be described by an approximation procedure. If E and F are subspaces of L^1_{loc}, then the following observation often enables us to conclude that every $[S] \in \mathcal{O}_M(E,F)$ acts by pointwise a.e. multiplication on all elements of E. (Observe that under the above hypothesis $\mathcal{O}_M(E,F) \subset L^1_{loc}$ by (3.3).)

(3.17) For all $\eta \in L^0(\Omega)$ the linear map $[\eta]: L^0(\Omega) \longrightarrow L^0(\Omega)$, $[\eta](f) := \eta \cdot f$, has closed graph and is thus continuous.

In more general situations we shall need a concept to multiply two individual distributions. There is a vast amount of articles on this topic. We only mention KÖNIG[39], BREMERMANN, DURAND[3], FUCHSSTEINER[22], KELLER[38], GONZÁLEZ DOMÍNGUEZ[24], AMBROSE[2], KAMIŃSKI[37] and COLOMBEAU[7].

Here we use a concept which was introduced by ITANO[35, p. 161]. A sequence $(\rho_k; k \in \mathbb{N})$ will be called a special δ-sequence if there exists $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varphi \geq 0$, $\|\varphi\|_1 = 1$, and a sequence $(\lambda_k; k \in \mathbb{N})$ of positive numbers tending to zero such that $\rho_k(x) = \lambda_k^{-n} \varphi(x/\lambda_k)$ $(x \in \mathbb{R}^n,\ k \in \mathbb{N})$.

(3.18) Definition (ITANO[35, p. 161]). Let $S, T \in \mathcal{D}(\Omega)'$. The multiplicative product $S \odot T$ of S and T exists if for every special δ-sequence $(\rho_k; k \in \mathbb{N})$ the sequence $(S \cdot (T * \rho_k); k \in \mathbb{N})$ is $\sigma(\mathcal{D}', \mathcal{D})$-convergent and the limit does not depend on the particular choice of the special δ-sequence. If these conditions are satisfied, we put

$$S \odot T := \lim_{k \to \infty} S \cdot (T * \rho_k).$$

For $E \subset \mathcal{D}(\Omega)'$ we define

$$E^M := \{T \in \mathcal{D}(\Omega)';\ S \odot T \text{ exists for all } S \in E\}.$$

If $\Omega \neq \mathbb{R}^n$, the above definition needs some explanation. Let K be a compact subset of Ω. Since $\operatorname{supp}(\rho_k) \to \{0\}$ $(k \to \infty)$, there exists $k_0 = k_0(K) \in \mathbb{N}$ such that $K - \operatorname{supp}(\rho_k) \subset \Omega$ holds for all $k \geq k_0$. On $\mathcal{D}_K(\Omega)$ the linear form $T * \rho_k$ is then defined by $\langle T * \rho_k, \psi \rangle := \langle T, \check{\rho}_k * \psi \rangle$ $(\psi \in \mathcal{D}_K(\Omega),\ k \geq k_0)$, and is actually generated by the $\mathcal{E}(\mathring{K})$-function $T * \rho_k(x) := \langle T, \rho_k(x - \hat{t}) \rangle$ $(x \in \mathring{K}, k \geq k_0)$. Thus, although $T * \rho_k$ is not defined on all of Ω we can investigate the convergence of the sequences $\langle S \cdot (T * \rho_k), \psi \rangle = \langle S, (T * \rho_k)\psi \rangle$ $(\psi \in \mathcal{D}_K(\Omega),\ k \geq k_0(K))$ for all $K \Subset \Omega$.

We refer to ITANO[35] for two other equivalent formulations of (3.18) as well as a throughout discussion of the product defined therein.

In view of (3.2) we restrict our investigation to the pointwise action of operators $[S] \in \mathcal{O}_M(E,\mathcal{D}')$ on elements of E, i.e. we only consider the special case $F := \mathcal{D}(\Omega)'$. For $\Omega = \mathbb{R}^n$ the next proposition was proved by ITANO[35, p.172].

<u>(3.19) Proposition.</u> For every normal barrelled space E of distributions on Ω we have

$$E^M \subset \mathcal{O}_M(E,\mathcal{D}') \quad \text{and} \quad [S](T) = S \odot T \quad (S \in E^M,\ T \in E).$$

<u>Proof.</u> Let $S \in E^M$. It follows immediately from (3.18) that $S \odot \varphi = S \cdot \varphi$ holds for all $\varphi \in \mathcal{D}(\Omega)$. Since $(\mathcal{D}(\Omega)', \beta(\mathcal{D}',\mathcal{D}))$ is the projective limit of the spaces $(\mathcal{D}_K(\Omega)', \beta(\mathcal{D}_K', \mathcal{D}_K))$ $(K \Subset \Omega)$ (cf. FLORET, WLOKA[20, p. 145]), it suffices to prove the continuity of $T \longmapsto S \odot T$ as a map from E to $\mathcal{D}_K(\Omega)'$ for ever $K \Subset \Omega$. Let $(\rho_k; k \in \mathbb{N})$ be a fixed special δ-sequence and let $K \Subset \Omega$ be given. Then there exists $L \Subset \Omega$ such that $K \subset \mathring{L}$. For $x \in \mathring{L}$ and $k \geq k_o(L)$ the map $x \longmapsto T * \rho_k(x) := \langle T, \rho_k(x-\hat{t}) \rangle$ is defined and infinitely differentiable. Moreover, the map $T \longmapsto T * \rho_k$ is continuous from $\mathcal{D}(\Omega)'$ to $\mathcal{E}(\mathring{L})$. Therefore the maps $T \longmapsto S \cdot (T * \rho_k)$ $(k \geq k_o(L))$ are continuous from $\mathcal{D}(\Omega)'$ into $\mathcal{D}_K(\Omega)'$, thus in particular from E into $\mathcal{D}_K(\Omega)'$. Now the barrelledness of E implies that the pointwise limit $T \longmapsto S \odot T = \lim_{k \to \infty} S \cdot (T * \rho_k)$ is continuous as a map from E into $\mathcal{D}_K(\Omega)'$.

<u>(3.20)</u> Under suitable hypotheses on E also the converse inclusion can be proved. For instance, if $\Omega = \mathbb{R}^n$ and E has the approximation property with respect to cutting and regularization, we have $\mathcal{O}_M(E,\mathcal{D}') \subset E^M$ (SHIRAISHI, ITANO[64, p. 232]).

To describe the corresponding situation for convolution operators we first recall the definition of convolution for distributions of L. SCHWARTZ. Let $\mathring{\mathcal{B}}(\Omega) := \{\varphi \in \mathcal{E}(\Omega);\ \partial^\alpha \varphi \in C_o(\Omega)\ (\alpha \in \mathbb{N}_o^n)\}$ be provided with its natural FRÉCHET space topology. The dual $\mathring{\mathcal{B}}(\mathbb{R}^n)'$ is the space of so called <u>integrable distributions</u>, and is denoted $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ in SCHWARTZ[60, p. 200]. It can be shown that the bidual $\mathring{\mathcal{B}}(\mathbb{R}^n)''$ is in a natural way isomorphic to the space $\mathcal{B}(\mathbb{R}^n) := \{\varphi \in \mathcal{E}(\mathbb{R}^n);\ \partial^\alpha \varphi \in L^\infty(\mathbb{R}^n)\ (\alpha \in \mathbb{N}_o^n)\}$. Therefore an <u>integral</u> can be defined for the elements $T \in \mathring{\mathcal{B}}(\mathbb{R}^n)'$ by

$$\int_{\mathbb{R}^n} T := {}_{\mathring{\mathcal{B}}'}\langle T, 1 \rangle_{\mathring{\mathcal{B}}''} \quad (T \in \mathring{\mathcal{B}}(\mathbb{R}^n)').$$

We refer to SCHWARTZ[58, exposé n° 21] and DIEROLF, VOIGT[13] for details. Observe that the above notions are a direct generalization of the notion of an integrable measure (HORVÁTH[31, p. 345]). The same is true for the following definition.

(3.21) Definition (SCHWARTZ[58, exposé n° 22],[59, p. 131]). Two distributions $S,T \in \mathcal{D}(\mathbb{R}^n)'$ are convolvable if for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have $\varphi(\hat{x}+\hat{y})(S\otimes T) \in \mathring{\mathcal{B}}(\mathbb{R}^{2n})'$. If this condition is satisfied, the convolution $S*T \in \mathcal{D}(\mathbb{R}^n)'$ is defined by

$$\langle S*T,\varphi\rangle := \int_{\mathbb{R}^n\times\mathbb{R}^n} S(x)\otimes T(y)\cdot\varphi(x+y)dxdy = \langle\varphi(\hat{x}+\hat{y})\cdot(S\otimes T),1\rangle$$

$(\varphi\in\mathcal{D}(\mathbb{R}^n))$. For $E\subset\mathcal{D}(\mathbb{R}^n)'$ we define

$$E^C := \{T\in\mathcal{D}(\mathbb{R}^n)';\ S*T \text{ exists for all } S\in E\}.$$

We refer to HORVÁTH[32], SHIRAISHI[62], ROIDER[55] and DIEROLF, VOIGT[12] for a detailed treatment of this notion. Here we only mention that SHIRAISHI[62] proved - among other results - that Definition (3.21) is equivalent to the definition of CHEVALLEY which was mentioned in (2.7).

The relation between E^C and $\mathcal{O}_C'(E,\mathcal{D}')$ is described in the following theorem which is more satisfactory than (3.19).

(3.22) Theorem. Let E be a normal barrelled space of distributions on $\mathbb{R}^n$. Then

(a) $E^C\subset\mathcal{O}_C'(E,\mathcal{D}')$ and $\{S\}(T) = S*T$ $(S\in E^C,\ T\in E)$.

(b) The inclusion $(E')^{\vee}\subset E^C$ implies $\mathcal{O}_C'(E,\mathcal{D}') = E^C =$ $= \{S\in\mathcal{D}(\mathbb{R}^n)';\ \check{S}*\varphi\in E'\ \ (\varphi\in\mathcal{D}(\mathbb{R}^n))\}$.

Proof. (a) is proved in YOSHINAGA, OGATA[78, p. 20, Thm. 3 (2)].
(b) By Thm. 2 of HIRATA[28, p. 92] we obtain $E^C = \{S\in\mathcal{D}(\mathbb{R}^n)';$ $\check{S}*\varphi\in E'\ (\varphi\in\mathcal{D}(\mathbb{R}^n))\}$. For $S\in\mathcal{O}_C'(E,\mathcal{D}')$ we have $\check{S}*\varphi\in E'$ $(\varphi\in\mathcal{D}(\mathbb{R}^n))$ by (3.4). Thus $\mathcal{O}_C'(E,\mathcal{D}')\subset E^C$, and the converse inclusion follows from (a).

We mention that for most of the usual spaces of distributions E on $\mathbb{R}^n$ the "C-dual" E^C was determined by YOSHINAGA, OGATA[78, p. 22, Thm. 5].

The FOURIER exchange formula.

If two tempered distributions $S,T\in\mathcal{S}(\mathbb{R}^n)'$ are convolvable in the sense of (3.21), there is no reason for the distribution $S*T$ to be tempered. Therefore, if one wants to extend the classical FOURIER exchange formula $\mathcal{F}(f*g) = \mathcal{F}(f)\cdot\mathcal{F}(g)$ $(f,g\in L^1)$ to tempered distributions, one needs a notion of convolvability which is more restrictive than (3.21). The following definition was first given by HIRATA, OGATA[29, p. 148] in a formulation similar to (2.7) (cf. also [12]).

(3.23) Definition (SHIRAISHI[62, p. 26/28]). Two tempered distributions $S,T \in \mathcal{S}(\mathbb{R}^n)'$ are $\mathcal{S}'$-convolvable if for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have $\varphi(\hat{x}+\hat{y})(S\otimes T) \in \mathring{\mathcal{B}}(\mathbb{R}^{2n})'$. If this condition is satisfied, the $\mathcal{S}'$-convolution $S \circledast T \in \mathcal{S}(\mathbb{R}^n)'$ is defined by

$$\langle S\circledast T, \varphi\rangle := \langle \varphi(\hat{x}+\hat{y})(S\otimes T), 1\rangle \quad (\varphi \in \mathcal{S}(\mathbb{R}^n)).$$

Now a generalization of the FOURIER exchange formula is given by the following proposition.

(3.24) Proposition. Let $S,T \in \mathcal{S}(\mathbb{R}^n)'$ be $\mathcal{S}'$-convolvable. Then the product $\mathcal{F}(S) \odot \mathcal{F}(T)$ exists in the sense of (3.18), and we have $\mathcal{F}(S\circledast T) = \mathcal{F}(S) \odot \mathcal{F}(T)$.

The above proposition was first proved by HIRATA, OGATA[29, p.151], where even a more restrictive notion of the product was used (cf. also SHIRAISHI, ITANO[64, p. 230, Rem. 2]).

In some cases, Proposition (3.24) can be used to connect the spaces $\mathcal{O}_C'(E,F)$ and $\mathcal{O}_M(\mathcal{F}(E),\mathcal{F}(F))$ as indicated in (2.1). It would however be desirable to have simple notions of $\mathcal{S}'$-convolution and $\mathcal{S}'$-product adapted to each other in such a way that also the converse of (3.24) holds, i.e. the existence of $S \odot T$ in an $\mathcal{S}'$-sense should imply the existence of $\mathcal{F}(S)\tilde{\ast}\mathcal{F}(T)$ in an $\mathcal{S}'$-sense. Several versions of such a theorem were recently given by KAMIŃSKI [37, p. 90].

Topological problems.

Here we only indicate by some examples the type of problems we have in mind. On the one hand the spaces $\mathcal{O}_M(E,F)$ and $\mathcal{O}_C'(E,F)$ are subspaces of $L(E,F)$. Thus there are (at least) two natural topologies on $\mathcal{O}_M(E,F)$ and $\mathcal{O}_C'(E,F)$, namely $\mathcal{T}_s \cap \mathcal{O}_M(E,F)$ and $\mathcal{T}_b \cap \mathcal{O}_M(E,F)$, where $\mathcal{T}_s$ and $\mathcal{T}_b$ denote the topology of $L_s(E,F)$ and $L_b(E,F)$, respectively. On the other hand $\mathcal{O}_M(E,F)$ and $\mathcal{O}_C'(E,F)$ are spaces of distributions and as such they usually have a natural locally convex topology $\mathcal{T}$, which is suggested by their structure. Moreover, this structure usually reflects in a strong sense the structure of the spaces E and F. For instance, we may represent $\mathcal{S}(\mathbb{R}^n)$ as the intersection of a decreasing sequence $(H_p;\, p\in\mathbb{N})$ of BANACH spaces, $\mathcal{S}(\mathbb{R}^n) = \bigcap_{p\in\mathbb{N}} H_p$. We then have $\mathcal{O}_M(\mathcal{S}) = \mathcal{O}_M(\bigcap_{p\in\mathbb{N}} H_p) = \bigcap_{q\in\mathbb{N}} \bigcup_{p\in\mathbb{N}} \mathcal{O}_M(H_p,H_q)$. Since $\mathcal{O}_M(H_p,H_q)$ has a natural BANACH space topology, the topology $\mathcal{T}$ suggested by the above representation is the topology of a countable projective limit of LB-spaces.(This is also true for the representation of $\mathcal{O}_M(\mathcal{S})$ which follows from (3.12).) It was shown by GROTHENDIECK[25, Part II, p. 130 ff] that $\mathcal{T}_s \cap \mathcal{O}_M(\mathcal{S}) = \mathcal{T}_b \cap \mathcal{O}_M(\mathcal{S}) =: \mathcal{R}$, and that $(\mathcal{O}_M(\mathcal{S}), \mathcal{R})$ is a complete nuclear bornological space whose dual is a nuclear LF-space. SHAW-YING TIEN[61] proved that $\mathcal{R}$ coincides with the above defined topology $\mathcal{T}$. The corresponding result for $\mathcal{O}_C'(\mathcal{K}_1')$ was proved by ZIELEŹNY[81]. According to GROTHENDIECK[25, Part II, p. 98], the space $L_b(\mathcal{E}, \mathcal{E})$ is a rather queer one. Nevertheless, on the subspace $\mathcal{E} = \mathcal{O}_M(\mathcal{E})$ the topologies $\mathcal{T}_b$ and $\mathcal{T}_s$ both induce the natural FRÉCHET space topology of $\mathcal{E}(\Omega)$.

On the other hand, it can be shown that $\mathcal{T}_b \cap \mathcal{O}_M(\mathring{\mathcal{B}}(\Omega)) \supsetneqq \mathcal{T}_s \cap \mathcal{O}_M(\mathring{\mathcal{B}}(\Omega))$ if Ω is not quasibounded (cf.[13, p. 78/79]).

4. CALCULATION OF SOME EXAMPLES

(4.1) Let $\emptyset \neq \Omega = \mathring{\Omega} \subset \mathbb{R}^n$, $p,q \in [1,\infty)$, and put $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then

$$\mathcal{O}_M(L^p,L^q) = \begin{cases} \{0\} & \text{for } q > p \\ L^r(\Omega) & \text{for } q \leq p \end{cases}, \text{ and } [h](f) = h\cdot f \quad (f\in L^p, h\in \mathcal{O}_M(L^p,L^q)).$$

Moreover $\mathcal{O}_M(L^p_{comp}, L^q_{loc}) = \mathcal{O}_M(L^p, L^q_{loc}) = \mathcal{O}_M(L^p_{loc}, L^q_{loc}) = L^r_{loc}(\Omega)$ for $q \leq p$.

Proof. By (3.3) we have $\mathcal{O}_M(L^p,L^q) \subset L^q_{loc}(\Omega)$, whereas (3.17) implies $\mathcal{O}_M(L^p,L^q) = \{h \in L^q_{loc}(\Omega);\ h\cdot f \in L^q(\Omega)\ (f \in L^p(\Omega))\}$ and $[h](f) = h\cdot f$ $(f\in L^p,\ h \in \mathcal{O}_M(L^p,L^q))$. If there exists $h \in \mathcal{O}_M(L^p,L^q) \setminus \{0\}$, we obtain $L^p(K) \subset L^q(K)$ for some $K \Subset \Omega$ with $\mathrm{meas}(K) > 0$, and thus $q \leq p$. For $q \leq p$, the inclusion $L^r(\Omega) \subset \mathcal{O}_M(L^p,L^q)$ follows from HÖLDER's inequality. If $p = q$, an appeal to LEBESGUE's differentiation theorem (STEIN[65, p. 5]) shows $|h(x)|^p \leq \|[h]\|^p$ a.e. on Ω, i.e. $\mathcal{O}_M(L^p) = L^\infty$. Now let $q < p$, $h \in \mathcal{O}_M(L^p,L^q)$ and put $\frac{1}{s} = 1 - \frac{1}{q}$, $\frac{1}{t} = \frac{1}{p} + \frac{1}{s}$. Then we have $h\cdot f\cdot g \in L^1$ for all $f \in L^p$, $g \in L^s$. Since the map $L^p \times L^s \longrightarrow L^t$, $(f,g) \longmapsto f\cdot g$, is surjective, we obtain $h\cdot k \in L^1$ for all $k \in L^t$, which implies $h \in (L^t)' = L^r$. The proof of the other equalities is similar.

(4.2) A suitable modification of the proof given in (3.12) shows

$$\mathcal{O}_M(\mathcal{S}, L^1) = \bigcup_{m\in\mathbb{N}} L^1(\mathbb{R}^n, \omega_{-m}) = \{h \in L^1_{loc};\ h\cdot\varphi \in L^1 \quad (\varphi \in \mathcal{S})\}$$

and

$$\mathcal{O}_M(\mathcal{S},\mathcal{M}^1) = \bigcup_{m\in\mathbb{N}} \mathcal{M}^1(\mathbb{R}^n,\omega_{-m}) = \{\mu\in\mathcal{M};\ \mu\cdot\varphi\in\mathcal{M}^1\quad(\varphi\in\mathcal{S})\}$$

(cf. also [9, p. 8]).

(4.3) Let $1\leq q\leq p<\infty$ and let $0<M_1\leq M_2\leq\ldots$ and $0<N_1\leq N_2\leq\ldots$ be two sequences in $\mathcal{E}^0(\Omega)$. We provide $E:=\bigcup_{j\in\mathbb{N}} L^p(\Omega,M_j^{-1})$ and $F:=\bigcup_{k\in\mathbb{N}} L^q(\Omega,N_k^{-1})$ with their natural LB-space topologies. Then

$$\begin{aligned}\mathcal{O}_M(E,F) &= \bigcap_{j\in\mathbb{N}}\bigcup_{k\in\mathbb{N}} \mathcal{O}_M(L^p(\Omega,M_j^{-1}),L^q(\Omega,N_k^{-1}))\\ &= \bigcap_{j\in\mathbb{N}}\bigcup_{k\in\mathbb{N}} L^r(\Omega,M_j\cdot N_k^{-1})\quad(\tfrac{1}{r}=\tfrac{1}{q}-\tfrac{1}{p}).\end{aligned}$$

In particular, for $S_{ar}(\mathbb{R}^n):=\mathcal{O}_M(\mathcal{S},L^1)$ we obtain $\mathcal{O}_M(S_{ar}(\mathbb{R}^n)) = \bigcup_{k\in\mathbb{N}} L^\infty(\mathbb{R}^n,\omega_{-k})$.

Proof. As in (4.1) we conclude $\mathcal{O}_M(E,F)\subset L^1_{loc}(\Omega)$ and $\mathcal{O}_M(E,F) = \{h\in L^1_{loc};\ h\cdot f\in F\ (f\in E)\}$. Let $h\in\mathcal{O}_M(E,F)$ be given. By Thm. A of GROTHENDIECK[25, Part I, p. 16], for every $j\in\mathbb{N}$ there exists $k\in\mathbb{N}$ such that $h\cdot f\in L^q(\Omega,N_k^{-1})$ for all $f\in L^p(\Omega,M_j^{-1})$, i.e. $h\cdot(g\cdot M_j)\cdot N_k^{-1}\in L^q(\Omega)$ for all $g\in L^p(\Omega)$. By (4.1) this implies $h\cdot M_j\cdot N_k^{-1}\in L^r(\Omega)$. The other inclusion is a direct consequence of the definition of the topologies on E and F.

(4.4) Let $\Omega=\mathring{\Omega}\subset\mathbb{R}^n$, $r(x):=\operatorname{dist}(x,\partial\Omega)$ if $\Omega\neq\mathbb{R}^n$, $r(x):=1$ if $\Omega=\mathbb{R}^n$, and put $\rho(x):=\min\{r(x),1\}$, $\rho_k(x):=(\rho(x))^k$ $(x\in\Omega,\ k\in\mathbb{Z})$. Let $\mathring{\mathcal{B}}(\Omega)=\{\varphi\in\mathcal{E}(\Omega);\ \partial^\alpha\varphi\in C_o(\Omega)\quad(\alpha\in\mathbb{N}_o^n)\}$ be provided with its natural FRÊCHET space topology. Then we have

$$\mathcal{O}_M(\mathring{\mathcal{B}}(\Omega)) = \{\psi\in\mathcal{E}(\Omega);\ \forall\alpha\in\mathbb{N}_o^n\ \exists m\in\mathbb{N}:\ \rho_m\partial^\alpha\psi\in L^\infty(\Omega)\}$$

and

$$\mathcal{O}_M(\mathring{\mathcal{B}},L^1) = \bigcup_{m\in\mathbb{N}} L^1(\Omega,\rho_m) = \{h\in L^1_{loc};\ h\cdot\varphi\in L^1\quad(\varphi\in\mathring{\mathcal{B}}(\Omega))\}.$$

(Observe that $\mathcal{O}_M(\mathring{\mathcal{B}}(\mathbb{R}^n),L^1(\mathbb{R}^n)) = L^1(\mathbb{R}^n)$ whereas $\mathcal{O}_M(\mathring{\mathcal{B}}(\Omega),L^1(\Omega))\supsetneqq L^1(\Omega)$ for $\Omega\neq\mathbb{R}^n$.)

Proof. We only prove the first equality, since this proof can be modified to yield the second statement. Moreover, since the results for $\Omega=\mathbb{R}^n$ are nearly obvious, we only treat the case $\Omega\neq\mathbb{R}^n$. We need some results from [13, Sect. 4]. It follows from [13, Prop. (4.6)] that $\mathring{\mathcal{B}}(\Omega)=\{\varphi\in\mathcal{E}(\Omega);\ \rho_{-m}\cdot\partial^\alpha\varphi\in C_o(\Omega)\quad(\alpha\in\mathbb{N}_o^n, m\in\mathbb{N}_o)\}$ holds, and that the semi-norms $\varphi\longmapsto\|\rho_{-m}\partial^\alpha\varphi\|_\infty$ $(m\in\mathbb{N}_o,\alpha\in\mathbb{N}_o^n)$ generate the natural FRÊCHET space topology on $\mathring{\mathcal{B}}(\Omega)$. Moreover, by [13, (4.3)] there exists a sequence $(\eta_k;k\in\mathbb{N})$ in $\mathcal{D}(\Omega)$ such that

(a) $\eta_k \geq 0 \quad (k \in \mathbb{N})$.

(b) $\forall K \Subset \Omega \ \exists k(K) \in \mathbb{N} \ \forall x \in K, \ k \geq k(K): \ \eta_k(x) = 1$.

(c) $\forall \alpha \in \mathbb{N}_0^n \ \exists C_\alpha > 0 \ \forall x \in \Omega, \ k \in \mathbb{N}: \ |\partial^\alpha \eta_k(x)| \leq C_\alpha r(x)^{-|\alpha|}$.

hold. Using the above representation of $\mathring{\mathcal{B}}(\Omega)$ we obtain $\{\ldots\} \subset \mathcal{O}_M(\mathring{\mathcal{B}}(\Omega))$. Now let $\psi \in \mathcal{O}_M(\mathring{\mathcal{B}}(\Omega))$ be given. Then $\psi \in \mathcal{E}(\Omega)$, and $[\psi](\varphi) = \psi \cdot \varphi \quad (\varphi \in \mathring{\mathcal{B}}(\Omega))$. We first choose $\xi_1 \in \mathcal{B}(\mathbb{R}^n)$ such that $\xi_1(x) = 1 \quad (r(x) \geq 1)$ and $\xi_1(x) = 0 \quad (r(x) < 1/2)$ hold. Then $\varphi_k := \eta_k \cdot (\xi_1 | \Omega) \quad (k \in \mathbb{N})$ is a bounded sequence in $\mathring{\mathcal{B}}(\Omega)$. Thus there exists $C_1 > 0$ such that $|\psi(x) \cdot \varphi_k(x)| \leq C_1 \quad (x \in \Omega, k \in \mathbb{N})$ holds. This implies $|\psi(x)| \leq C_1$ for all $x \in \Omega$ satisfying $r(x) \geq 1$. Now we choose $\xi_2 \in \mathcal{B}(\mathbb{R}^n)$ such that $\xi_2(x) = 1 \quad (r(x) \leq 1)$, $\xi_2(x) = 0 \quad (r(x) > 3/2)$ hold. Since $\|\cdot\|_\infty$ is a continuous norm on $\mathring{\mathcal{B}}(\Omega)$ there exist $m \in \mathbb{N}$ and $C_2 > 0$ such that $\|\psi \cdot \varphi\|_\infty \leq$ $\leq C_2 \cdot \max\{\|\varrho_{-m} \partial^\alpha \varphi\|_\infty ; |\alpha| \leq m\} \quad (\varphi \in \mathring{\mathcal{B}}(\Omega))$ holds. By $\tilde{r} : \Omega \longrightarrow (0, \infty)$ we denote the regularized boundary distance (STEIN [65, p. 171, Thm. 2]). We define $\varphi_k := (\xi_2 | \Omega) \cdot \eta_k \cdot \tilde{r}^m \quad (k \in \mathbb{N})$. Using the estimate (c) for $(\eta_k ; k \in \mathbb{N})$ and the estimates for $\tilde{r}$ (STEIN, loc. cit), we obtain $\sup\{\|\varrho_{-m} \cdot \partial^\alpha \varphi_k\|_\infty ; |\alpha| \leq m, \ k \in \mathbb{N}\} < \infty$, which shows $\psi \cdot (\xi_2 | \Omega) \cdot \tilde{r}^m \in L^\infty(\Omega)$. (To carry out these estimates, it is convenient to use the results of FRAENKEL[21] or [9, p. 126].) Taking the two estimates together we obtain $\varrho_m \cdot \psi \in L^\infty(\Omega)$. Now an appeal to (3.11) finishes the proof.

We would like to point out that the original description of $\mathring{\mathcal{B}}(\Omega)$ didn't give any hint for the weight functions $\varrho_m (m \in \mathbb{N})$ occuring in the description of $\mathcal{O}_M(\mathring{\mathcal{B}}(\Omega))$.

The above result can also be proved by a method similar to (3.12). One then uses the fact that to every bounded subset of $\bigcap_{m \in \mathbb{N}} L^\infty(\Omega, \varrho_{-m})$ there exists $\psi \in \mathring{\mathcal{B}}(\Omega)''$ (sic!) such that $|f(x)| \leq \psi(x)$ a.e. on Ω holds for all $f \in B$, and the inclusion $\mathcal{O}_M(\mathring{\mathcal{B}}(\Omega)) \subset \mathcal{O}_M(\mathring{\mathcal{B}}(\Omega)'')$.

Now we shall calculate some spaces of convolution operators. The following result is implicitly contained in SCHWARTZ[60, p. 192 ff].

<u>(4.5)</u> $\mathcal{O}_C'(L^1_{comp}, L^1_{loc}) = \mathcal{O}_C'(L^1_{comp}, \mathcal{M}) = \mathcal{O}_C'(\mathcal{K}, L^\infty_{loc}) = \mathcal{O}_C'(\mathcal{K}, \mathcal{E}^0) = \mathcal{M}(\mathbb{R}^n)$.

<u>Proof.</u> Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. Then $\{\mu\} : \mathcal{E}(\mathbb{R}^n)' \longrightarrow \mathcal{D}(\mathbb{R}^n)'$ is continuous and $\{\mu\}(f) = \mu * f$ in the sense of convolution of measures holds for all $f \in L^1_{comp}$. Therefore $\{\mu\}(L^1_{comp}) \subset L^1_{loc}$ (cf. DIEUDONNÉ

[15, p. 262]) and $\mu \in \mathcal{O}_C'(L^1_{comp}, L^1_{loc})$ follows by a closed graph theorem. We thus proved $\mathcal{M}(\mathbb{R}^n) \subset \mathcal{O}_C'(L^1_{comp}, L^1_{loc})$. The continuous inclusion $L^1_{loc} \hookrightarrow \mathcal{M}$ yields $\mathcal{O}_C'(L^1_{comp}, L^1_{loc}) \subset \mathcal{O}_C'(L^1_{comp}, \mathcal{M})$. Now let $T \in \mathcal{O}_C'(L^1_{comp}, \mathcal{M})$ be given. From $\mathcal{K} \subset \mathcal{O}_C'(\mathcal{M}, \mathcal{E}^0)$ we infer that for every $\psi \in \mathcal{K}$ the map $f \longmapsto (T * f) * \psi = (T * \psi) * f$ is continuous from L^1_{comp} into $\mathcal{E}^0$. Therefore $f \longmapsto (T * \psi) * \check{f}(0) = \langle T * \psi, f \rangle$ is a continuous linear form on L^1_{comp}, i.e. $T * \psi \in (L^1_{comp})' = L^\infty_{loc}$ for all $\psi \in \mathcal{K}$. This shows $\mathcal{O}_C'(L^1_{comp}, \mathcal{M}) \subset \mathcal{O}_C'(\mathcal{K}, L^\infty_{loc})$. Let $S \in \mathcal{O}_C'(\mathcal{K}, L^\infty_{loc})$. A simple approximation argument shows $S * \psi \in \mathcal{E}^0$ for all $\psi \in \mathcal{K}$, which in turn implies $S \in \mathcal{O}_C'(\mathcal{K}, \mathcal{E}^0)$, i.e. $\mathcal{O}_C'(\mathcal{K}, L^\infty_{loc}) \subset \mathcal{O}_C'(\mathcal{K}, \mathcal{E}^0)$. Finally we obtain $\mathcal{O}_C'(\mathcal{K}, \mathcal{E}^0) \subset (\mathcal{K}')^\vee =$ $= \mathcal{M}$ by (3.6).

<u>(4.6)</u> We provide the space $C_b(\mathbb{R}^n) := \mathcal{E}^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with the strict topology γ which is generated by the semi-norms $f \mapsto \|f \cdot \varphi\|_\infty$ ($\varphi \in C_0(\mathbb{R}^n)$). We note that $\gamma = \tau(C_b, \mathcal{M}^1)$ (CONWAY[8, p. 478, Thm. 2.6]). The spaces $\mathcal{M}^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ are provided with the MACKEY topologies $\tau(\mathcal{M}^1, C_0)$ and $\tau(L^\infty, L^1)$, respectively. We have

$$\mathcal{O}_C'(L^1) = \mathcal{O}_C'(C_0) = \mathcal{O}_C'(\mathcal{M}^1) = \mathcal{O}_C'(C_b) = \mathcal{O}_C'(L^\infty) = \mathcal{M}^1(\mathbb{R}^n)$$

and

$$\mathcal{O}_C'(L^1_{comp}, L^p) = \mathcal{O}_C'(L^1, L^p) = L^p(\mathbb{R}^n) \quad (p \in (1, \infty]).$$

<u>Proof.</u> By DIEUDONNÉ[15, p. 265] we obtain $\mathcal{M}^1 \subset \mathcal{O}_C'(C_0)$ and $\mathcal{M}^1 \subset \mathcal{O}_C'(L^1)$. By (3.4) and (3.6) we thus obtain $\mathcal{M}^1 \subset \mathcal{O}_C'(C_0) = \mathcal{O}_C'(\mathcal{M}^1)$ and $\mathcal{O}_C'(C_0) \subset (C_0')^\vee = \mathcal{M}^1$. Therefore $\mathcal{M}^1 = \mathcal{O}_C'(C_0) = \mathcal{O}_C'(\mathcal{M}^1)$. Now let $T \in \mathcal{O}_C'(L^1)$. Then there exists $C > 0$ such that $\|T * \varphi\|_v \leq C \cdot \|\varphi\|_v = C \cdot \|\varphi\|_1$ ($\varphi \in \mathcal{D}(\mathbb{R}^n)$), where $\|\cdot\|_v$ denotes the variation norm on $\mathcal{M}^1$. Let $(\rho_k; k \in \mathbb{N})$ be a special δ-sequence as described before (3.18). Then $T * \rho_k \longrightarrow T$ $(k \to \infty)$ in $\mathcal{D}(\mathbb{R}^n)'$ and $\|\rho_k\|_1 = 1$ $(k \in \mathbb{N})$. Since $\{\mu \in \mathcal{M}^1; \|\mu\|_v \leq C\}$ is $\sigma(\mathcal{M}^1, C_0)$-compact, we thus obtain $T \in \mathcal{M}^1$. Therefore $\mathcal{O}_C'(L^1) = \mathcal{O}_C'(L^\infty) = \mathcal{M}^1(\mathbb{R}^n)$. The proof of $\mathcal{O}_C'(L^1_{comp}, L^p) = \mathcal{O}_C'(L^1, L^p) = L^p(\mathbb{R}^n)$ $(p \in (1, \infty])$ is similar. (We note in passing that the above method can also be used to prove that even the $\beta(\mathcal{M}^1, C_0)$-continuous "convolution operators" on $\mathcal{M}^1$ coincide with $\mathcal{M}^1$.)

By (3.6) we obtain $\mathcal{O}_C'(C_b) \subset (C_b')^\vee = \mathcal{M}^1$. Now let $\mu \in \mathcal{M}^1$. From

DIEUDONNÉ [15, p. 265] we infer $\mu * f \in C_b$ $(f \in C_b)$. Therefore it remains to show the continuity of $\{\mu\}: C_b \longrightarrow C_b$. Since C_b is a MACKEY space, it is sufficient to prove the $\sigma(C_b, \mathcal{M}^1) - \sigma(C_b, \mathcal{M}^1)$-continuity of $\{\mu\}$. Let $\nu \in \mathcal{M}^1$. By FUBINI's theorem we have $\langle \mu * f, \nu \rangle = \langle f, \check{\mu} * \nu \rangle$. Now $\check{\mu} * \nu \in \mathcal{M}^1$ implies the $\sigma(C_b, \mathcal{M}^1)$-continuity of $f \longmapsto \langle \mu * f, \nu \rangle$ $(\nu \in \mathcal{M}^1)$.

<u>(4.7)</u> We provide the space $\mathcal{B}(\mathbb{R}^n) = \mathring{\mathcal{B}}(\mathbb{R}^n)''$ (cf. [13]) with the MACKEY topology $\tau(\mathcal{B}, \mathring{\mathcal{B}}')$ (cf. [11]). Then

$$\mathcal{B}(\mathbb{R}^n)^C = \mathcal{O}_C'(\mathring{\mathcal{B}}, \mathcal{D}') = \mathcal{O}_C'(\mathcal{D}, \mathring{\mathcal{B}}') = \mathcal{O}_C'(\mathring{\mathcal{B}}) = \mathcal{O}_C'(\mathring{\mathcal{B}}') =$$
$$= \mathcal{O}_C'(\mathring{\mathcal{B}}, \mathcal{E}) = \mathcal{O}_C'(\mathcal{E}', \mathring{\mathcal{B}}') = \mathcal{O}_C'(\mathcal{B}) = \mathcal{O}_C'(\mathcal{D}, L^1) = \mathring{\mathcal{B}}'.$$

<u>Proof.</u> Since $\mathring{\mathcal{B}}$ is a normal barrelled space we obtain $\mathring{\mathcal{B}}^C \subset \mathcal{O}_C'(\mathring{\mathcal{B}}, \mathcal{D}')$ by (3.22a). From $\mathcal{M}^1 \subset \mathcal{O}_C'(C_o)$ we obtain $\partial^\alpha \mu \in \mathcal{O}_C'(\mathring{\mathcal{B}})$ $(\alpha \in \mathbb{N}_o^n, \mu \in \mathcal{M}^1)$, and thus $\mathring{\mathcal{B}}' \subset \mathcal{O}_C'(\mathring{\mathcal{B}}) \subset \mathring{\mathcal{B}}^C$ by the representation theorem for $\mathring{\mathcal{B}}(\mathbb{R}^n)'$ (HORVÁTH[31, p. 347]). Thus (3.22b) implies $\mathring{\mathcal{B}}^C = \mathcal{O}_C'(\mathring{\mathcal{B}}, \mathcal{D}') = \{S \in \mathcal{D}(\mathbb{R}^n)'; \check{S} * \varphi \in \mathring{\mathcal{B}}'$ $(\varphi \in \mathcal{D})\} = \mathcal{O}_C'(\mathcal{D}, \mathring{\mathcal{B}}')$. If $S \in \mathcal{O}_C'(\mathcal{D}, \mathring{\mathcal{B}}')$ we can use the parametrix method (cf. (3.5)) to prove $S \in \mathring{\mathcal{B}}'$. Until now we have proved $\mathring{\mathcal{B}}' \subset \mathring{\mathcal{B}}^C = \mathcal{O}_C'(\mathring{\mathcal{B}}, \mathcal{D}') = \{\ldots\} = \mathcal{O}_C'(\mathcal{D}, \mathring{\mathcal{B}}') \subset \mathring{\mathcal{B}}'$. As noted above $\mathring{\mathcal{B}}' \subset \mathcal{O}_C'(\mathring{\mathcal{B}})$. By (3.1) and (3.6) we obtain $\mathcal{O}_C'(\mathring{\mathcal{B}}) \subset \mathcal{O}_C'(\mathring{\mathcal{B}}, \mathcal{E}) \subset \mathring{\mathcal{B}}'$, which implies $\mathring{\mathcal{B}}' = \mathcal{O}_C'(\mathring{\mathcal{B}}) = \mathcal{O}_C'(\mathring{\mathcal{B}}_\tau') = \mathcal{O}_C'(\mathring{\mathcal{B}}, \mathcal{E}) = \mathcal{O}_C'(\mathcal{E}', \mathring{\mathcal{B}}')$, whereas $\mathcal{O}_C'(\mathring{\mathcal{B}}_\beta') \subset \mathcal{O}_C'(\mathcal{B}) \subset \mathcal{B}' = \mathring{\mathcal{B}}'$ follows from (3.4) and (3.6). Thus $\mathring{\mathcal{B}}' \subset \mathcal{O}_C'(\mathring{\mathcal{B}}_\tau') \subset \mathcal{O}_C'(\mathring{\mathcal{B}}_\beta') = \mathcal{O}_C'(\mathcal{B}) \subset \mathring{\mathcal{B}}'$. By the above mentioned representation theorem for $\mathring{\mathcal{B}}'$, (4.6), and the parametrix method we obtain the following characterization

$T \in \mathring{\mathcal{B}}' \Longleftrightarrow T * \varphi \in L^1$ $(\varphi \in \mathcal{D}) \Longleftrightarrow \exists\, m \in \mathbb{N}$, $(f_\alpha; |\alpha| \leq m)$ in L^1 such that $T = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha$, which finally yields $\mathcal{O}_C'(\mathcal{D}, L^1) = \mathring{\mathcal{B}}'$.

<u>5. FURTHER RESULTS.</u>

In this section we mention some other results which could not be presented in detail.

<u>(5.1)</u> Characterizations of the spaces $\mathcal{O}_M(L_p^m, L_p^l)$ and $\mathcal{O}_M(W^{m,p}, W^{m,l})$ $(m \geq 1,\ p \in (1, \infty))$ on $\mathbb{R}^n$ by means of capacities have been proved by MAZJA, ŠAPOŠNIKOVA[48](cf. also [46 , 47]). Earlier results are due to STRICHARTZ[69]. Multiplication operators on spaces of BESOV-HARDY-SOBOLEV type were investigated by TRIEBEL[72, 73].

<u>(5.2)</u> The spaces $\mathcal{O}_M(\mathcal{K}_1')$ and $\mathcal{O}_C'(\mathcal{K}_1')$ and their topologies were

characterized by YOSHINAGA[77] and ZIELEŹNY[81] (cf. also HASUMI[26]) whereas the spaces $\mathcal{O}'_C(\mathcal{K}'_p)$ $(p > 1)$ were characterized by SAMPSON, ZIELEŹNY[56].

(5.3) In a recent paper POORNIMA[53] showed $\mathcal{O}'_C(W^{m,p}) = \mathcal{O}'_C(L^p)$, $\mathcal{O}'_C(L^1, W^{m,p}) = W^{m,p}$ $(p \in (1, \infty))$, $\mathcal{O}'_C(L^1, W^{m,1}) = \{f \in L^1;\ \partial^\alpha f \in \mathcal{M}^1$ $(|\alpha| \leq m)\}$, and proved some results on $\mathcal{O}'_C(W^{m,1}, L^1)$ and $\mathcal{O}'_C(W^{m,1})$. Convolution operators on spaces of BESOV-HARDY-SOBOLEV type were investigated by TRIEBEL[74].

(5.4) In [9] and [10] we studied the space $S_r = \mathcal{S}' \cap L^1_{loc}$ of regular tempered distributions on $\mathbb{R}^n$. The topology of this space is the LF-space topology given by $S_r = \bigcup_{m \in \mathbb{N}_o} (H_{-m} \cap L^1_{loc})$, where $\mathcal{S}' = \bigcup_{m \in \mathbb{N}_o} H_{-m}$ is a representation of $\mathcal{S}'$ as an inductive limit of HILBERT spaces (cf. KUČERA[40]). The following result is proved in [9, p. 55]:

$$\mathcal{O}_M(S_r) = \bigcap_{q \in \mathbb{N}_o} \bigcup_{p \in \mathbb{N}_o} (\mathcal{O}_M(H_{-q}, H_{-p}) + L^\infty_{comp}) =$$
$$= \{h \in L^\infty_{loc};\ \forall\, m \in \mathbb{N}\ \exists\, k \in \mathbb{N},\ h_1 \in \mathcal{S}^m_{-k},\ h_2 \in L^\infty_{comp}:\ h = h_1 + h_2\}.$$

(5.5) The following interesting approach was invented by FIGÀ-TALAMANCA[19] and RIEFFEL[54]. Most spaces of distributions are multiplication modules and/or convolution modules over the algebra $\mathcal{D}$. But then the multiplication (convolution) operators are exactly the continuous module homomorphisms, i.e. $\mathcal{O}_M(E,F) = \mathrm{Hom}_{\mathcal{D}}(E,F)$ and $\mathcal{O}'_C(E,F) = \mathrm{Hom}_{\mathcal{D}}(E,F)$, where the module operations are multiplication and convolution, respectively. Every representation theorem for $\mathrm{Hom}_{\mathcal{D}}(E,F)$, e.g. of the type $\mathrm{Hom}_A(E,F') = (E \otimes_A F)'$, which is valid for BANACH-A-modules, gives a representation of $\mathcal{O}_M(E,F)$ $(\mathcal{O}'_C(E,F))$ as the dual of some module tensor product. There is a good chance to realize these duals again as spaces of distributions on Ω $(\mathbb{R}^n)$. This program has been carried out by RIEFFEL[54] for BANACH-$L^1(G)$-modules over a locally compact group G. To our knowledge a corresponding theory for spaces of distributions has not been worked out up to now.

We close this paper with the following remark. Our impression is that the simpler the structure of the spaces E and F, the harder is the computation of the spaces $\mathcal{O}_M(E,F)$ and $\mathcal{O}'_C(E,F)$. (The example (4.1) is exceptional.) If for instance E and F are F-spaces or LF-spaces, the gap between necessary and sufficient conditions for $T\in\mathcal{O}_M(E,F)$ in terms of the step spaces can often be "pushed to infinity".

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STRUCTURE OF CLOSED LINEAR TRANSLATION INVARIANT SUBSPACES OF A($\mathbb{C}$) AND KERNELS OF ANALYTIC CONVOLUTION OPERATORS

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Let A($\mathbb{C}$) denote the vector space of all entire functions on $\mathbb{C}$, endowed with the compact-open topology. Every continuous linear functional μ on A($\mathbb{C}$) induces a continuous linear map T_μ on A($\mathbb{C}$) by

$$T_\mu(f) : z \mapsto \langle \mu_w, f(z+w) \rangle , \quad f \in A(\mathbb{C}).$$

These operators are called convolution operators and can also be regarded as differential operators of infinite order with constant coefficients. From this point of view, the structure of ker T_μ has already been investigated by Ritt [18] in 1917. A first answer to the more general question about the structure of the closed linear translation invariant subspaces of A($\mathbb{C}$) was given by Schwartz [19]. Concerning the representation of the elements of ker T_μ by exponential monomials, Gelfond [9], Dickson [4] and Ehrenpreis [8] showed that, for every convolution operator T_μ on A($\mathbb{C}$), ker T_μ has a finite dimensional decomposition for which the finite dimensional blocks are spanned by exponential polynomials.

The aim of the present note is to report on some progress concerning the study of such questions which has been made by the work of Berenstein and Taylor [1],[2], Taylor [21], the author [15] and Meise and Schwerdtfeger [16]. Even though the results are rather general, we restrict our attention here to the special situation introduced above since this allows a clear exposition of the ideas without too many technicalities. For a more general survey on part of this work (up to 1980) we refer to the article of Berenstein and Taylor [3].

This report is divided in three sections. In the first one, we introduce the convolution operators on A($\mathbb{C}$) and show that the question on the structure of the closed linear translation invariant subspaces of A($\mathbb{C}$) is equivalent - up to duality - to the structure of the quotients of the space Exp($\mathbb{C}$) of entire functions of exponential type by its closed ideals. In section 2 we explain how a result of Schwartz [19] on closed ideals in Exp($\mathbb{C}$) and the minimum modulus theorem, together with the approach of Berenstein and Taylor [1], lead to a fairly explicit model for Exp($\mathbb{C}$)/I, where I is a closed non-zero ideal with infinite codimension. Then we describe in section 3 how an observation of the author [15] can be used to derive from this model the following result: Every infinite dimensional closed linear proper subspace W of $A(\mathbb{C})$ which is translation invariant has a Schauder basis consisting of exponential polynomials. With respect to this basis W is isomorphic to a nuclear power series space of infinite type. W is a complemented subspace of $A(\mathbb{C})$. Since this applies, in particular, to the kernels of convolution operators, it shows that the finite dimensional decomposition of ker T_μ mentioned above actually comes from grouping a certain Schauder basis. The model of $\mathrm{Exp}(\mathbb{C})/I$ obtained so far is then applied to derive, for a certain class of convolution operators T_μ, a necessary condition that the exponential monomials form a basis of ker T_μ. Concluding we use this condition to get some examples.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

1.1 CONVOLUTION OPERATORS ON $A(\mathbb{C})$

By $A(\mathbb{C})$ we denote the space of all entire functions on $\mathbb{C}$, endowed with the usual compact-open topology. The strong dual of $A(\mathbb{C})$ will be denoted by $A(\mathbb{C})_b'$; its elements will be called analytic functionals.

If μ is an analytic functional, then it is easy to check that μ induces a continuous linear operator T_μ on $A(\mathbb{C})$ by the following definition

(1) $$T_\mu(f) : z \mapsto \langle \mu_w, f(z+w) \rangle , \quad z \in \mathbb{C}, \ f \in A(\mathbb{C}).$$

These operators are called convolution operators. They can also be characterized as those continuous linear operators on $A(\mathbb{C})$ which commute with all the translation operators $\tau_a : f \mapsto f(\cdot+a)$, $a \in \mathbb{C}$. If $f \in A(\mathbb{C})$ has the Taylor expansion $f(z) = \sum_{n=o}^{\infty} f_n z^n$ and if we put $\mu_n := \langle \mu, z^n \rangle$, $n \in \mathbb{N}_o$, then one can show that, for all $z \in \mathbb{C}$,

(2) $$T_\mu(f)[z] = \sum_{n=o}^{\infty} \frac{1}{n!} \mu_n f^{(n)}(z) = \sum_{k=o}^{\infty} \Big(\sum_{n=o}^{\infty} \frac{(n+k)!}{n!k!} f_{n+k} \mu_n \Big) z^k .$$

Hence every convolution operator can be regarded as a differential operator of infinite order with constant coefficients.

1.2 THE CONVOLUTION ALGEBRA $(A(\mathbb{C})_b', *)$ AND THE FOURIER-BOREL ISOMORPHISM

If μ and ν are analytic functionals, we define their convolution product $\mu * \nu \in A(\mathbb{C})'$ by

(1) $$\langle \mu*\nu, f \rangle := \sum_{n=o}^{\infty} f_n \Big(\sum_{k+j=n} \frac{n!}{k!j!} \mu_k \nu_j \Big),$$

where $f(z) = \sum_{n=o}^{\infty} f_n z^n$ and $\mu_n := \langle \mu, z^n \rangle$, $\nu_n := \langle \nu, z^n \rangle$. It is easy to check that $(A(\mathbb{C})_b', *)$ is a commutative locally convex algebra with unit and that

(2) $$< \mu * \nu, f >_{A(\mathbb{C})} = < \mu \otimes \nu, f^{\Delta} >_{A(\mathbb{C}^2)},$$

where $f^{\Delta} : (z,w) \mapsto f(z+w)$.

In order to remark that the algebra $A(\mathbb{C})_b'$ is isomorphic to an algebra of functions, we put

$$\mathrm{Exp}(\mathbb{C}) = \{f \in A(\mathbb{C}) \mid \text{there exists } A > 0 \text{ with } \sup_{z\in\mathbb{C}} |f(z)| e^{-A|z|} < \infty\}$$

and endow $\mathrm{Exp}(\mathbb{C})$ with its natural inductive limit topology. It is easy to see that $\mathrm{Exp}(\mathbb{C})$ is a commutative locally convex algebra with unit and that the "Fourier-Borel" map $F : A(\mathbb{C})_b' \to \mathrm{Exp}(\mathbb{C})$, defined by

(3) $$F(\mu)[z] := < \mu_w, e^{zw} > , \ z \in \mathbb{C}, \ \mu \in A(\mathbb{C})_b',$$

is a topological algebra isomorphism. Obviously we have for all $z \in \mathbb{C}$

(4) $$F(\mu)[z] = \sum_{n=0}^{\infty} \frac{1}{n!} \mu_n z^n.$$

1.3 THE PROBLEM

Our aim is to get a satisfactory description of the kernel of a given convolution operator T_μ. In the special case that T_μ is a differential operator, everybody knows how to do this. In the more general situation, the same method also provides certain elements in $\ker T_\mu$. We introduce the following notation:

If $\mu \in A(\mathbb{C})'$, $\mu \neq 0$, is given, then we put $V(\mu) := \{a \in \mathbb{C} \mid F(\mu)[a] = 0\}$.

For $a \in V(\mu)$, we denote by m_a the multiplicity of the zero a of $F(\mu)$. Hence we have $F(\mu)^{(j)}(a) = 0$ for $0 \le j < m_a$ and $F(\mu)^{(m_a)}(a) \neq 0$. For $a \in V(\mu)$ and $0 \le j < m_a$ we denote by $E_{j,a}$ the so-called exponential monomials

$$E_{j,a} : z \mapsto z^j e^{az}.$$

From 1.2(3), we get for all $z \in \mathbb{C}$ and all $k \in \mathbb{N}_0$

(1) $$F(\mu)^{(k)}[z] = < \mu_w, w^k e^{zw} > .$$

This implies for $a \in V(\mu)$ and $0 \le j < m_a$ that

$$\begin{aligned} T_\mu(E_{j,a})[z] &= < \mu_w, (z+w)^j e^{a(z+w)} > \\ &= \sum_{k=0}^{j} \binom{j}{k} z^{k-j} e^{az} < \mu_w, w^k e^{aw} > \\ &= \sum_{k=0}^{j} \binom{j}{k} z^{k-j} e^{az} F(\mu)^{(k)}[a] = 0 \end{aligned}$$

and hence $E_{j,a} \in \ker T_\mu$. Note that

$$E := \{E_{j,a} \mid a \in V(\mu), \ 0 \le j < m_a\}$$

is a free set. The functions in span(E) are called exponential solutions (or exponential polynomials) of the convolution operator T_μ.
We remark that, for every $f \in \text{span}(E)$ (resp. $\ker T_\mu$) and every $a \in \mathbb{C}$, the function $\tau_a(f)$ is again in span (E) (resp. $\ker T_\mu$), i.e. span(E) and $\ker T_\mu$ are translation invariant linear subspaces of $A(\mathbb{C})$. Hence we have the following two natural questions:

(a) Is it possible to obtain all elements of $\ker T_\mu$ from the exponential monomials by a certain procedure?

Or more generally:

(b) How can one describe the structure of the closed linear translation invariant subspaces of $A(\mathbb{C})$?

It is classical to attack these questions by applying duality theory to get a different interpretation. As the first observation, we note that every convolution operator is the adjoint of a multiplication operator.

<u>1.4 LEMMA.</u> *For* $\mu \in A(\mathbb{C})'$ *define* $M_\mu : A(\mathbb{C})'_b \to A(\mathbb{C})'_b$ *by* $M_\mu(\nu) := \mu * \nu$. *Then* ${}^tM_\mu = T_\mu$ *if we identify* $(A(\mathbb{C})'_b)'$ *with* $A(\mathbb{C})$.

<u>PROOF.</u> For $j \in \mathbb{N}_0$, let $\varepsilon(j)$ denote the analytic functional satisfying $< \varepsilon(j), z^n > = \delta_{j,n}$ for all $n \in \mathbb{N}_0$. Then it follows from 1.2(1) that for $f : z \to \sum_{n=0}^{\infty} f_n z^n$ we have

$$< {}^tM_\mu(f), \varepsilon(j) > = < f, M_\mu(\varepsilon(j)) > = < f, \mu * \varepsilon(j) > =$$

$$= \sum_{n=j}^{\infty} f_n \frac{n!}{j!(n-j)!} \mu_{n-j} = \sum_{n=0}^{\infty} f_{n+j} \frac{(n+j)!}{j!n!} \mu_n .$$

By 1.1(2), this implies

$${}^tM_\mu(f)[z] = \sum_{k=0}^{\infty} < {}^tM_\mu(f), \varepsilon(k) > z^k$$

$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(n+k)!}{n!k!} f_{n+k} \mu_n \right) z^k = T_\mu(f)[z],$$

and hence ${}^tM_\mu = T_\mu$.

<u>1.5 PROPOSITION.</u> *Let* W *be a closed linear subspace of* $A(\mathbb{C})$. *Then* W *is translation invariant if and only if* $W^\perp$ *is an ideal in the convolution algebra* $(A(\mathbb{C})'_b, *)$.

<u>*PROOF.*</u> Assume that W is translation invariant. Since, for every $f \in A(\mathbb{C})$, we have $\lim_{h \to 0} \frac{f(\cdot+h)-f(\cdot)}{h} = f'$ in the topology of $A(\mathbb{C})$, we see that $f \in W$ implies that

$f^{(n)}$ is in W for all $n \in \mathbb{N}$. Hence we get for each $\mu \in W^{\perp}$ and each $f \in W$ with $f(z) = \sum_{n=0}^{\infty} f_n z^n$, that for all $n \in \mathbb{N}_0$,

$$(1) \qquad 0 = \langle \mu, f^{(n)} \rangle = \langle \mu, \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} f_{n+k} z^k \rangle = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} f_{n+k} \mu_k .$$

By 1.2(1), it follows that, for each $\nu \in A(\mathbb{C})'$,

$$\langle \mu * \nu, f \rangle = \sum_{n=0}^{\infty} f_n \Big(\sum_{k+j=n} \frac{n!}{k!j!} \mu_k \nu_j \Big)$$

$$(2) \qquad = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (k+j)! f_{k+j} \frac{\mu_k}{k!} \frac{\nu_j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{\nu_j}{j!} \Big(\sum_{k=0}^{\infty} \frac{(k+j)!}{k!} f_{k+j} \mu_k \Big) = 0.$$

Hence $W^{\perp}$ is a closed ideal in $(A(\mathbb{C})_b', *)$.

To prove the converse, let us assume that $W^{\perp}$ is an ideal in $A(\mathbb{C})$. Then we get from 1.4 that, for all $\mu \in W^{\perp}$, all $f \in W$ and all $a \in \mathbb{C}$,

$$0 = \langle \mu * \delta_a, f \rangle = \langle M_{\delta_a}(\mu), f \rangle = \langle \mu, T_{\delta_a}(f) \rangle = \langle \mu, \tau_a(f) \rangle .$$

Hence $\tau_a(f)$ is in $W^{\perp\perp} = W$ for all $a \in \mathbb{C}$, i.e. W is translation invariant.

1.6 REFORMULATION OF THE PROBLEM

Since $A(\mathbb{C})$ is a nuclear Fréchet space, well-known duality results show that, for each closed linear subspace W of $A(\mathbb{C})$,

$$(1) \qquad W = W^{\perp\perp} \simeq (A(\mathbb{C})_b' / W^{\perp})_b' .$$

Applying the Fourier-Borel isomorphism, we get

$$(2) \qquad W \simeq (\mathrm{Exp}(\mathbb{C}) / \mathcal{F}(W^{\perp}))_b' .$$

Hence it follows from 1.5 that - up to the computation of a dual space - question 1.3(b) is equivalent to determining the quotient of the algebra $\mathrm{Exp}(\mathbb{C})$ by a closed ideal. In case that $W = \ker T_{\mu}$, it follows from 1.4 that $\mathcal{F}(W^{\perp}) = \mathcal{F}(\overline{\mathrm{Im}(M_{\mu})})$, which is the closure of the principal ideal $\mathcal{F}(\mu) \cdot \mathrm{Exp}(\mathbb{C})$.

In the next section, we shall see that this reformulation has the advantage that we can apply results from complex analysis to study the quotient spaces $\mathrm{Exp}(\mathbb{C})/I$, where I is a closed ideal in $\mathrm{Exp}(\mathbb{C})$.

2. THE MAIN TOOLS

In order to derive a fairly explicit description of $\mathrm{Exp}(\mathbb{C})/I$ for all closed ideals I in $\mathrm{Exp}(\mathbb{C})$, we use some (rather special) properties of the ideals and the elements of Exp($\mathbb{C}$) which we will now recall.

2.1 LOCALIZATION OF IDEALS

For $a \in \mathbb{C}$, we denote by $\mathcal{O}_a$ the ring of germs of holomorphic functions at a. If I is an ideal in Exp($\mathbb{C}$), then I_a denotes the ideal generated by the canonical image of I in $\mathcal{O}_a$. The localization I_{loc} of the ideal I is defined as

$$I_{loc} = \{f \in \mathrm{Exp}(\mathbb{C}) \mid [f]_a \in I_a \text{ for all } a \in \mathbb{C}\},$$

where $[f]_a$ denotes the germ of f at a. It is easy to check that I_{loc} is a closed ideal in Exp($\mathbb{C}$) which contains I.
Since every non-zero ideal in $\mathcal{O}_a$ is of the form $[(z-a)]_a^{m_a} I_a$ for a suitable $m_a \in \mathbb{N}_0$, the non-zero localized ideals $I = I_{loc}$ are completely determined by the set $V(I) := \{a \in \mathbb{C} \mid m_a > 0\}$ and the numbers m_a, $a \in V(I)$. As an example, let us look at the ideal $I(f_1,\dots,f_n)$ generated by $f_1,\dots,f_n$ in Exp($\mathbb{C}$). For its localization $I_{loc}(f_1,\dots,f_n)$, it is easy to see that

$$V(I_{loc}(f_1,\dots,f_n)) = \{a \in \mathbb{C} \mid f_j(a) = 0 \text{ for } 1 \le j \le n\}$$

and that m_a equals the minimum of the order of the zeros of the functions f_j at a.

In the following theorem we state the special property of the ideals in Exp($\mathbb{C}$) which we are going to use. The theorem is due to Schwartz [19] and Ehrenpreis [7]; for a proof, we refer to [7], sect. 6, or to Kelleher and Taylor [11], where rather general extensions of this result are presented.

2.2 THEOREM. *a) Every closed ideal in* Exp($\mathbb{C}$) *is localized.*
b) For every closed ideal I *in* Exp($\mathbb{C}$), *there exist* $f_1, f_2 \in \mathrm{Exp}(\mathbb{C})$ *such that* $I = I_{loc}(f_1,f_2)$.

Besides Theorem 2.2, we shall use the following property of the non-zero functions in Exp($\mathbb{C}$), which can be derived from the minimum modulus theorem (see e.g. Levin [13],I,§8).

2.3 PROPOSITION. *For every* $f \in \mathrm{Exp}(\mathbb{C})$, $f \neq 0$, *there exist* $\varepsilon > 0$, $C > 0$ *and* $(r_n)_{n\in\mathbb{N}}$ *with* $2^n < r_n < 2^{n+1}$ *such that, for all large* $n \in \mathbb{N}$, *we have*

$$\inf_{t\in[0,2\pi]} |f(r_n e^{it})| \ge \varepsilon e^{-Cr_n}.$$

Using some functional analysis, it is easy to conclude, from this property, the following classical result on analytic convolution operators, due to Ehrenpreis [7] and Malgrange [14]; it already gives a partial answer to 1.3(a).

2.4 PROPOSITION. *a) Every principal ideal in* Exp($\mathbb{C}$) *is closed.*
b) Every non-zero convolution operator T_μ *is surjective, and the exponential*

solutions of T_μ *are dense in ker* T_μ.

Proof. a) We may assume $I = f\,\mathrm{Exp}(\mathbb{C})$, where $f \neq 0$. Since $I \subset I_{loc}$ and since I_{loc} is closed, it suffices to show $I_{loc} \subset I$; to prove this, let $g \in I_{loc}$ be given. The considerations in 2.1 show that $g = h \cdot f$ for some $h \in A(\mathbb{C})$. Obviously, it suffices to show $h \in \mathrm{Exp}(\mathbb{C})$. At this point, we remark that, from 2.3 and $g \in \mathrm{Exp}(\mathbb{C})$, we obtain positive numbers D and M such that

$$\sup_{t \in [0,2\pi]} |h(r_n e^{it})|\, e^{-Dr_n} \leq M \text{ for all large } n \in \mathbb{N}\,.$$

This and the estimate on r_n given in 2.3, together with an application of the maximum principle to the annulus $\{z \in \mathbb{C} \mid r_n \leq |z| \leq r_{n+1}\}$, show that $h \in \mathrm{Exp}(\mathbb{C})$.

b) Since the Fourier-Borel transform is a topological algebra isomorphism, we get from 1.4 and part a) that ${}^tT_\mu = M_\mu$ is injective and that $\mathrm{im}\,{}^tT_\mu = \mathrm{im}\, M_\mu$ is closed. Hence the surjectivity of T_μ is a consequence of a classical result of Dieudonné and Schwartz (see Horvâth [10],p. 308).

By the Hahn-Banach theorem, the linear subspace E of the exponential solutions of T_μ is dense in ker T_μ iff $E^\perp = (\mathrm{kern}\, T_\mu)^\perp$. Since $\mathrm{im}\,{}^tT_\mu = \mathrm{im}\, M_\mu$ is closed, it suffices to prove $E^\perp \subset \mathrm{im}\, M_\mu$; so let $\nu \in E^\perp$. Then, for all $a \in V(\mu) = V(F(\mu))$ and $0 \leq j < m_a$,

$$0 = \langle \nu, E_{j,a} \rangle = \langle \nu, z^j e^{az} \rangle = \sum_{n=0}^{\infty} \nu_{n+j} \frac{a^n}{n!} = \sum_{n=0}^{\infty} \frac{(n+j)!}{n!} \frac{\nu_{n+j}}{(n+j)!} a^n$$

$$= F(\nu)^{(j)}(a),$$

and this shows $F(\nu) \in I_{loc}(F(\mu)\mathrm{Exp}(\mathbb{C})) = F(\mu)\mathrm{Exp}(\mathbb{C})$. Hence there exists $\lambda \in A(\mathbb{C})'$ with $\nu = \mu * \lambda = M_\mu(\lambda)$, i.e. $\nu \in \mathrm{im}\, M_\mu = E^\perp$.

Now we are ready to sketch how to obtain a fairly explicit model for $\mathrm{Exp}(\mathbb{C})/I$, following the approach of Berenstein and Taylor [1]; it suffices to consider the closed non-zero ideals I of $\mathrm{Exp}(\mathbb{C})$ which are of infinite codimension.

2.5 THE STRUCTURE OF $\mathrm{Exp}(\mathbb{C})/I$

Let I be a closed ideal in $\mathrm{Exp}(\mathbb{C})$ which is different from $\{0\}$ and $\mathrm{Exp}(\mathbb{C})$. We define $\rho : \mathrm{Exp}(\mathbb{C}) \to \prod_{a \in V(I)} \mathcal{O}_a/I_a$ by $\rho(f) := ([f]_a + I_a)_{a \in V(I)}$. It is easy to see that

$\ker \rho = I_{loc}$ and since $I = I_{loc}$ by 2.2, this gives $\ker \rho = I$. At this point the structure of $\mathrm{Exp}(\mathbb{C})/I$ will be clear if im ρ, as a locally convex space, is described in such a way that $\rho : \mathrm{Exp}(\mathbb{C}) \to \mathrm{im}\,\rho$ is a topological homomorphism. We will do this in several steps.

(1) The slowly decreasing property

By 2.2b), we have $I = I_{loc}(f_1, f_2)$, where we can assume $f_1 \neq 0$. In view of 2.4b), we shall assume from now on that $V(I)$ is infinite. Because of 2.3, we can choose $\varepsilon > 0$, $C > 0$ and $B > 0$ such that

(i) each component S of

$$S(f_1,f_2;\varepsilon,C) := \{z \in \mathbb{C} \mid |f_1(z)|^2 + |f_2(z)|^2 < (\varepsilon \exp(-C|z|))^2\}$$

is bounded and satisfies $\operatorname{diam} S \leq B \sup_{z\in S}|z|$ and,

(ii) such that for each component S of $S(f_1,f_2;\varepsilon,C)$,

$$\sup_{z\in S}|z| \leq B(\inf_{z\in S}|z|) + B.$$

As we shall see later, this is the appropriate extension of the slowly decreasing condition of Berenstein and Taylor [1], p. 130.

(2) Labeling the components of $S(f_1,f_2;\varepsilon,C)$

By our assumption, $V(I)$ is an infinite discrete subset of $\mathbb{C}$ contained in $S(f_1,f_2;\varepsilon,C)$. Hence (i) of (1) implies that $S(f_1,f_2;\varepsilon,C)$ has infinitely many components S with $S \cap V(I) \neq \emptyset$. We label these components by natural numbers in such a way that the sequence α, defined by $\alpha_j := \sup_{z\in S_j} |z|$, is non-decreasing.

(3) The Banach spaces $(E_j, \| \ \|_j)$

Let $A^\infty(S_j)$ denote the space of all bounded holomorphic functions on S_j, endowed with the norm $\| \ \| : f \mapsto \sup_{z\in S_j} |f(z)|$. Put $E_j := \prod_{a\in S_j \cap V(I)} \mathcal{O}_a/I_a$ and define $\rho_j : A^\infty(S_j) \to E_j$ by $\rho_j(g) := ([g]_a + I_a)_{a\in S_j\cap V(I)}$. It is easy to see that ρ_j is surjective. Hence we can endow E_j with the corresponding quotient norm, i.e. with the norm

$$\| \ \|_j : \varphi \mapsto \inf\{\|g\|_{A^\infty(S_j)} \mid \rho_j(g) = \varphi\}.$$

(4) The spaces $k(\gamma,\mathbb{F})$

Let $\mathbb{F} = (F_j, \| \ \|_j)_{j\in\mathbb{N}}$ be a sequence of Banach spaces and let γ denote an increasing unbounded sequence of non-negative real numbers. Then we define

$$k(\gamma,\mathbb{F}) := \{x \in \prod_{j=1}^{\infty} F_j \mid \text{there exists } A > 0 \text{ such that } \sup_{j\in\mathbb{N}} \|x_j\|_j e^{-A\gamma_j} < \infty\}$$

and endow $k(\gamma,\mathbb{F})$ with its natural inductive limit topology.
If $\dim F_j < \infty$ for all $j \in \mathbb{N}$, then it is easy to check that $k(\gamma,\mathbb{F})$ is a (DFS)-space, i.e. the strong dual of a Fréchet-Schwartz space.

(5) The semi-local interpolation theorem

Let g denote a holomorphic function on $S(f_1,f_2;\varepsilon,C)$ such that, for some $B > 0$,

$\sup\{|g(z)|e^{-B|z|} \mid z \in S(f_1,f_2;\varepsilon,C)\} < \infty$. Then there exists $G \in \text{Exp}(\mathbb{C})$ with $[g]_a - [G]_a \in I_a$ for all $a \in V(I)$.
For a proof of this result we refer to Berenstein and Taylor [1], p. 120.

(6) The map $\rho : \text{Exp}(\mathbb{C}) \to k(\alpha,\mathbb{E})$

Let $\mathbb{E} = (E_j,\| \|_j)_{j\in\mathbb{N}}$ denote the sequence of finite dimensional Banach spaces introduced in (3), and let α denote the sequence introduced in (2). If, for some positive numbers A and D, $f \in \text{Exp}(\mathbb{C})$ satisfies the estimate $|f(z)| \leq Ae^{D|z|}$ for all $z \in \mathbb{C}$, then, by the definition of α,

$$\|f|S_j\|_{A^\infty(S_j)} \leq Ae^{D\alpha_j} \text{ for all } j \in \mathbb{N}\text{, whence}$$

$$\sup_{j\in\mathbb{N}} \|\rho_j(f|S_j)\|_j e^{-D\alpha_j} \leq A.$$

At this point the map ρ defined at the beginning can be considered as a map of $\text{Exp}(\mathbb{C})$ into $k(\alpha,\mathbb{E})$, given by $\rho(f) = (\rho_j(f|S_j))_{j\in\mathbb{N}}$. Moreover, the above estimates show that ρ is continuous. Since $\text{Exp}(\mathbb{C})$ and $k(\alpha,\mathbb{E})$ are (DFS)-spaces, the open mapping theorem for (LF)-spaces applies, and ρ is an open map iff it is surjective.
To prove the surjectivity of ρ, let $x = (x_j)_{j\in\mathbb{N}} \in k(\alpha,\mathbb{E})$ be given. Then there exist $A,D > 0$ with $\sup_{j\in\mathbb{N}} \|x_j\|_j e^{-D\alpha_j} \leq A$. By the definition of $\| \|_j$ and by (ii) of (1), this implies the existence of $g_j \in A^\infty(S_j)$ such that $\rho_j(g_j) = x_j$ and

$$\|g_j\|_{A^\infty(S_j)} \leq 2Ae^{D\alpha_j} \leq 2Ae^{DB}e^{D|z|} \text{ for all } z \in S_j.$$

Hence the function $g \in A(S(f_1,f_2;\varepsilon,C))$, defined by $g|S_j = g_j$ for $j \in \mathbb{N}$ and $g|S = 0$ for the components S of $S(f_1,f_2;\varepsilon,C)$ with $S \cap V(I) = \emptyset$, satisfies the hypotheses of the semi-local interpolation theorem (5) and, by (5), there is $G \in \text{Exp}(\mathbb{C})$ with $\rho(G) = x$.

Thus, we have already sketched the proof of the following result:

2.6 THEOREM. *Let* I *be a non-zero closed ideal of* $\text{Exp}(\mathbb{C})$ *with infinite codimension. Then* $\text{Exp}(\mathbb{C})/I$ *is isomorphic to* $k(\alpha,\mathbb{E})$.

In order to derive more information from Theorem 2.6, Berenstein and Taylor [1] used Newton interpolation to introduce equivalent norms $|||\ |||_j$ on the spaces E_j. In this way, they obtained a representation of $\text{Exp}(\mathbb{C})/I$ as a space of scalar sequences. But the norms $|||\ |||_j$ are computed by divided differences and, using this representation, it is difficult to discover special structural properties. That $\text{Exp}(\mathbb{C})/I$ really has a very special structure follows from a remark of the author [15] which will be described in the next section.

3. SOLUTION OF THE PROBLEM

If we want to derive the solution of the problem posed in 1.3 from the results presented in section 2, we need some more preparations.

3.1 POWER SERIES SPACES OF INFINITE TYPE

Let γ be an increasing unbounded sequence of non-negative real numbers, and let $\mathbb{F} = (F_j, \| \ \|_j)_{j\in\mathbb{N}}$ be a sequence of Banach spaces. For $1 \leq p < \infty$, we define the spaces $\Lambda^p_\infty(\gamma,\mathbb{F})$ by

$$\Lambda^p_\infty(\gamma,\mathbb{F}) := \{x \in \prod_{j=1}^{\infty} F_j \mid \pi_{r,p}(x) = (\sum_{j=1}^{\infty} (\|x_j\|_j e^{r\gamma_j})^p)^{1/p} < \infty \text{ for all } r \geq 0\}.$$

Obviously, $\Lambda^p_\infty(\gamma,\mathbb{F})$ is a Fréchet space under the canonical norm system $(\pi_{r,p})_{r\geq 0}$. If $(F_j, \| \ \|_j) = (\mathbb{C}, |\cdot|)$ for all $j \in \mathbb{N}$, then we write $\Lambda^p_\infty(\gamma)$ instead of $\Lambda^p_\infty(\gamma,\mathbb{F})$. $\Lambda^p_\infty(\gamma)$ is called a power series space of infinite type. We remark that, by the Grothendieck-Pietsch criterion (see Pietsch [17],6.1), $\Lambda^p_\infty(\gamma)$ is nuclear if and only if $\sup_{j\in\mathbb{N}} \frac{\log(j+1)}{\gamma_j} < \infty$. If $\Lambda^p_\infty(\gamma)$ is nuclear, then $\Lambda^p_\infty(\gamma) = \Lambda^q_\infty(\gamma)$ for all $p,q \in [1,\infty)$.

By the work of Dubinsky [5], Vogt [22],[23] and Vogt and Wagner [24],[25], the (stable) nuclear power series spaces of infinite type are a class of Fréchet spaces the structural properties of which are very well understood. We will make use of this fact later on.

3.2 CONSTRUCTION OF A SCHAUDER BASIS

Let $W \neq A(\mathbb{C})$ denote a closed linear subspace of $A(\mathbb{C})$ which is translation invariant and infinite dimensional. By 1.5 and 1.2, $I := F(W^\perp)$ is a closed ideal in $\mathrm{Exp}(\mathbb{C})$ of infinite codimension. By 2.2, $I = I_{loc}(f_1,f_2)$ for appropriate $f_1,f_2 \in \mathrm{Exp}(\mathbb{C}), f_1 \neq 0$. Hence all the hypotheses of 2.5 are satisfied. Using the notation introduced in 2.5, we define now, for all $a \in V(I)$ and $0 \leq k < m_a$, elements $e_{k,a} \in W$, $t_{k,a} \in I^\perp$ and $y_{k,a} \in k(\alpha,\mathbf{E})$ in the following way:

(1) $e_{k,a} : z \mapsto \frac{1}{k!} z^k e^{az}$, $z \in \mathbb{C}$

(2) $t_{k,a} : f \mapsto \frac{1}{k!} f^{(k)}(a)$, $f \in \mathrm{Exp}(\mathbb{C})$

(3) $y_{k,a}$ has germ 0 at every $b \in V(I)$, $b \neq a$, and at a it is the germ $[(z-a)^k]_a$.

Clearly, $t_{k,a}$ belongs to $I^\perp$. Since $F : A(\mathbb{C})' \to \mathrm{Exp}(\mathbb{C})$ is an isomorphism and $I = F(W^\perp)$, we have ${}^tF(I^\perp) = W^{\perp\perp} = W$. Because of

$$< {}^tF(t_{k,a}),\mu > = < t_{k,a},F(\mu) > = \frac{1}{k!} \sum_{n=0}^{\infty} \frac{1}{n!} \mu_{n+k} a^n = < e_{k,a},\mu >$$

for all $\mu \in A(\mathbb{C})'$, we get

$$(4) \qquad {}^tF(t_{k,a}) = e_{k,a},$$

and hence $e_{k,a} \in W$. Obviously $y_{k,a} \in E_j$ if $a \in V(I) \cap S_j$ and if we identify E_j with its canonical image in $k(\alpha,\mathbb{E})$.
Next, identifying $Exp(\mathbb{C})/I$ with $k(\alpha,\mathbb{E})$ (by the map given by Theorem 2.6) as well as $(Exp(\mathbb{C})/I)'$ with $I^{\perp}$, it is immediate that

$$(5) \qquad t_{1,b}(y_{k,a}) = \delta_{1,k}\delta_{a,b}.$$

In [15], it is shown that one can find a Hilbert norm $|\,|_j$ on E_j, $j \in \mathbb{N}$, such that, with $\tilde{\mathbb{E}} := (E_j,|\,|_j)_{j\in\mathbb{N}}$, the locally convex space $k(\alpha,\mathbb{E})$ is identical with

$$k^2(\alpha,\tilde{\mathbb{E}}) = \{x \in \prod_{j=1}^{\infty} E_j \mid \text{there exists } A>0 : (\sum(|x_j|_j e^{-A\alpha_j})^2)^{1/2} < \infty\}$$

under its natural inductive limit topology. Putting

$$F_j := \text{span}\{t_{k,a} \mid 0 \leq k < m_a,\ a \in V(I) \cap S_j\} \subset I^{\perp},$$

it follows from (5), and from the remark that $\{y_{k,a} \mid 0 \leq k < m_a,\ a \in V(I) \cap S_j\}$ is a basis of E_j, that F_j can be interpreted as the dual of the Hilbert space $(E_j,|\,|_j)$. If we denote by $|\,|_j$ the dual norm of $(E_j,|\,|_j)$, then $(F_j,|\,|_j)$ is a Hilbert space, too. Now let $\mathbb{F} := (F_j,|\,|_j)_{j\in\mathbb{N}}$; we remark that, by Theorem 2.6, the map

$$\Phi : \Lambda_{\infty}^2(\alpha,\mathbb{F}) \to Exp(\mathbb{C})_b',$$

defined by

$$\Phi((\xi_j)_{j\in\mathbb{N}})[f] := \sum_{j=1}^{\infty} < \xi_j, f > ,\ f \in Exp(\mathbb{C}),$$

gives an isomorphism between $\Lambda_{\infty}(\alpha,\mathbb{F})$ and $I^{\perp}$.
As we have explained in [15], one can now get a basis in $\Lambda_{\infty}^2(\alpha,\mathbb{F})$ in the following way: Choose an orthonormal basis $(h_{k,j})_{k=1}^{n_j}$ in $(F_j,|\,|_j)$ ($n_j := \dim F_j = \dim E_j$) for each $j \in \mathbb{N}$ and identify $h_{k,j}$ with its canonical image in $\Lambda_{\infty}^2(\alpha,\mathbb{F})$. Then $((h_{k,j})_{k=1}^{n_j})_{j\in\mathbb{N}}$ is an absolute basis in $\Lambda_{\infty}^2(\alpha,\mathbb{F})$. If we denote by β the sequence which is obtained by repeating each number α_j n_j-times and if we write the elements of $\Lambda_{\infty}^2(\beta)$ as $((\xi_{k,j})_{k=1}^{n_j})_{j\in\mathbb{N}}$, then the map $A : \Lambda_{\infty}^2(\beta) \to \Lambda_{\infty}^2(\alpha,\mathbb{F})$, defined by

$$A(((\xi_{k,j})_{k=1}^{n_j})_{j\in\mathbb{N}}) := \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \xi_{k,j} h_{k,j},$$

is an isomorphism.

Of course, we can assume that the orthonormal basis $(h_{k,j})_{k=1}^{n_j}$ of $(F_j,|\,|_j)$ is obtained from the basis $\{t_{k,a} \mid 0 \leq k < m_a,\ a \in V(I) \cap S_j\}$ by Gram-Schmidt orthonormalization. Since ${}^tF$ gives an isomorphism from $I^{\perp}$ to W and since (4) holds, we have sketched the complete proof of the following theorem answering question 1.3(b):

3.3 THEOREM. *Let* W *be a proper closed linear subspace of* A($\mathbb{C}$) *which is translation invariant and infinite dimensional. Then* W *has a Schauder basis consisting of exponential polynomials with respect to which* W *is isomorphic to a nuclear power series space of infinite type.*

In the special case $W = \ker T_\mu, \mu \neq 0$, the considerations in 3.2 give the following result which, up to a certain extent, answers question 1.3(a) and which improves previous representation theorems of Dickson [4], Gelfond [9] and Ehrenpreis [8] (see also Berenstein and Taylor [1], Thm. 9).

3.4 THEOREM. *Let* T_μ *be a non-zero convolution operator on* A($\mathbb{C}$) *for which* $\ker T_\mu$ *is infinite dimensional. Then there exist a partition* $(V_j)_{j\in\mathbb{N}}$ *of* $V(\mathcal{F}(\mu))$, *linear combinations* $f_{k,j}$, $1 \le k \le n_j := \sum_{a\in V_j} (m_a - 1)$, *of the functions* $\{z^l e^{az} \mid 0 \le l < m_a, a \in V_j\}$ *for each* $j \in \mathbb{N}$, *and an exponent sequence* α *such that the following holds:*

For every family $\xi = ((\xi_{k,j})_{k=1}^{n_j})_{j\in\mathbb{N}}$ *of complex numbers which satisfies* $\sum_{j=1}^{\infty} (\sum_{k=1}^{n_j} |\xi_{k,j}|) e^{r\alpha_j} < \infty$ *for all* $r > 0$, *the series*

$$(*) \qquad \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \xi_{k,j} f_{k,j}$$

converges normally to an element of $\ker T_\mu$, *and every* $f \in \ker T_\mu$ *has a unique representation of this type. In particular,* $\ker T_\mu$ *is isomorphic to a nuclear power series space of infinite type.*

To show that the structural properties derived so far have further implications, we now indicate how they can be used, together with the splitting theorem of Vogt [22] and an observation of [15], to give a new proof of the following result which, by 1.5, is equivalent to Taylor [21], thm. 5.1.

3.5 THEOREM. *Every closed linear translation invariant subspace* W *of* A($\mathbb{C}$) *is complemented. In particular, every non-zero convolution operator on* A($\mathbb{C}$) *has a complemented kernel.*

Sketch of the proof. If $W = \ker T_\mu$, where μ is a non-zero convolution operator, then W is complemented whenever $\dim W < \infty$. But if $\dim W = \infty$, then 2.4b) shows that we have the following exact sequence of Fréchet spaces

$$(*) \qquad 0 \longrightarrow W \xrightarrow{i} A(\mathbb{C}) \xrightarrow{T_\mu} A(\mathbb{C}) \longrightarrow 0.$$

By Theorem 3.4, W is a power series space of infinite type. Since $A(\mathbb{C}) \simeq \Lambda_\infty^1(n)$, the sequence (*) splits by Vogt [22], thm. 7.1; hence W is complemented. If W is an arbitrary translation invariant closed linear subspace, then we may assume $W \neq A(\mathbb{C})$ and $\dim W = \infty$. From 2.2b), it follows that there exist non-zero analytic functionals μ and ν with $W = (\ker T_\mu) \cap (\ker T_\nu)$, and consequently $W = \ker(T_\nu | \ker T_\mu)$. By the previous argument, $\ker T_\mu$ is complemented. Hence W is complemented if $\ker (T_\nu | \ker T_\mu)$ is complemented in $\ker T_\mu$. But this follows from the structure of $\ker T_\mu$ as described in 3.2 and an elementary lemma. For details, we refer to [15].

Remark. a) Results on the structure of the closed linear translation invariant subspaces (resp. the kernels of convolution operators) analogous to those given in the theorems 3.3, 3.4 and 3.5 can also be obtained for Frêchet spaces A of entire functions different from $A(\mathbb{C})$. This has been demonstrated in [15] and, more generally, in Meise and Schwerdtfeger [16].
b) A different proof of the fact that every non-zero convolution operator T_μ on $A(\mathbb{C})$ has a complemented kernel was given by Schwerdtfeger [20]. He used results of Gelfond [9] and Dickson [4] to show that $\ker T_\mu$ has property (Ω), which is sufficient for the application of the splitting theorem of Vogt [22].

The answer to question 1.3(a) which we have given in Theorem 3.4 is not yet complete, since it does not exclude that already the exponential monomials $\{z^k e^{az} \mid 0 \le k < m_a,\ a \in V(F(\mu))\}$ form a Schauder basis of $\ker T_\mu$ for every convolution operator T_μ on $A(\mathbb{C})$. However, as classical results of Leont'ev [12] indicate this is not true in general. To conclude, let us show now how the model of $\ker T_\mu$ obtained so far can be used to derive a simple necessary condition which leads to examples of convolution operators T_μ for which the exponential monomials do not form a Schauder basis of $\ker T_\mu$.

3.6 DEDUCTION OF A NECESSARY CONDITION

Let T_μ be a non-zero convolution operator on $A(\mathbb{C})$ for which $\ker T_\mu$ is infinite dimensional and for which the exponential monomials $\{z^k e^{az} \mid 0 \le k < m_a, a \in V(F(\mu))\}$ form a Schauder basis of $\ker T_\mu$. We put $W = \ker T_\mu$, $I = F(\mu)\mathrm{Exp}(\mathbb{C})$ and use the notation introduced in 2.5 and 3.2. In 3.2, we have indicated that, by ${}^tF$, $\ker T_\mu$ is isomorphic to $\Lambda^2_\infty(\alpha,\mathbb{F})$. Hence it follows from 3.2(4) that $\{t_{k,a} \mid 0 \le k < m_a, a \in V(F(\mu))\}$ is a Schauder basis of $\Lambda^2_\infty(\alpha,\mathbb{F})$. Again in 3.2 we have remarked that $k(\alpha,\mathbb{E}) = k^2(\alpha,\tilde{\mathbb{E}})$ and, by the same arguments, we get $\Lambda^2_\infty(\alpha,\mathbb{F}) = \Lambda^1_\infty(\alpha,\mathbb{E}')$, where $\mathbb{E}' = ((E_j, \| \ \|_j)'_b)_{j\in\mathbb{N}}$. It is easy to check that, by $\langle (x_j)_{j\in\mathbb{N}}, (y_j)_{j\in\mathbb{N}} \rangle := \sum_{j\in\mathbb{N}} \langle x_j, y_j \rangle_j$, the space $k(\alpha,\mathbb{E}) = k(\alpha,\mathbb{E}'')$ is the dual space of $\Lambda^1_\infty(\alpha,\mathbb{E}')$. Hence 3.2(5) implies that, with respect to this duality, the system $\{y_{k,a}\}$ is the system of coefficient functionals of the basis $\{t_{k,a}\}$.

We claim:

(1) There exists $D>0$ such that $\sup_{j\in\mathbb{N}} \sup_{a\in V_j} \sup_{0\le k<m_a} \|t_{k,a}\|_j' \|y_{k,a}\|_j e^{-D\alpha_j} < \infty$,

where $V_j := V(F(\mu)) \cap S_j$.

In order to prove this claim, we first remark that, because of the nuclearity of $\Lambda_\infty^2(\alpha,\mathbb{F}) = \Lambda_\infty^1(\alpha,\mathbb{E}')$ and the basis theorem of Dynin and Mityagin [6], $\{t_{k,a}\}$ is an absolute basis. Arguing by contradiction, we assume that (1) does not hold. Then, for every $n\in\mathbb{N}$ there exist $j(n), a(n) \in V_{j(n)}$ and $0 \le k(n) < m_{a(n)}$ with

$$\|t_{k(n),a(n)}\|_{j(n)}' \quad \|y_{k(n),a(n)}\|_{j(n)} \ge \exp(2n\alpha_{j(n)}).$$

Without loss of generality we can assume that $(j(n))_{n\in\mathbb{N}}$ is strictly increasing. Next, for each $n\in\mathbb{N}$, we choose $x_{j(n)} \in E_{j(n)}$ with $\|x_{j(n)}\|_{j(n)} = 1$ and $\|t_{k(n),a(n)}\|_{j(n)}' = t_{k(n),a(n)}(x_{j(n)})$. Then we define $y \in \prod_{j\in\mathbb{N}} E_j$ by $y_{j(n)} = x_{j(n)}\exp(-n\alpha_{j(n)})$ for all $n\in\mathbb{N}$ and $y_j = 0$ for all $j\in\mathbb{N}\setminus \bigcup_{n\in\mathbb{N}}\{j(n)\}$. It is easy to check that $y\in\Lambda_\infty^1(\alpha,\mathbb{E}')$. Since we have

$$|t_{k(n),a(n)}(y)|\ \pi_{0,1}(y_{k(n),a(n)}) = t_{k(n),a(n)}(y_{j(n)})\|y_{k(n),a(n)}\|_{j(n)}$$
$$= \|t_{k(n),a(n)}\|_{j(n)}'\|y_{k(n),a(n)}\|_{j(n)}\exp(-n\alpha_{j(n)}) \ge \exp(n\alpha_{j(n)})$$

for each $n\in\mathbb{N}$, the system $\{t_{k,a}\}$ cannot be an absolute basis.

Now let us assume that, in addition to the hypotheses made so far, we also have the following:

(2) There exists $E>0$ such that $\sup_{j\in\mathbb{N}} (\operatorname{diam} S_j)^{m_j} e^{-E\alpha_j} < \infty$, where $m_j := \max_{a\in V_j}(m_a - 1)$.

We note that, for the functions $\varphi_{k,a} : z \mapsto (z-a)^k$ $(0\le k<m_a, a\in V_j)$, we have

$$1 = t_{k,a}(\rho_j(\varphi_{k,a})) \le \|t_{k,a}\|_j' \|\rho_j(\varphi_{k,a})\|_j \le \|t_{k,a}\|_j' \|\varphi_{k,a}\|_{A^\infty(S_j)}$$
$$\le \|t_{k,a}\|_j' \max(1, (\operatorname{diam} S_j)^k).$$

Hence (2) implies the existence of $E>0$ with

$$\inf_{j\in\mathbb{N}} \inf_{a\in V_j} \inf_{0\le k<m_a} \|t_{k,a}\|_j' e^{E\alpha_j} > 0;$$

together with (1), this gives:

(3) There exists $F>0$ such that $\sup_{j\in\mathbb{N}} \sup_{a\in V_j} \sup_{0\le k<m_a} \|y_{k,a}\|_j e^{-F\alpha_j} < \infty$.

Hence $B := \{y_{k,a} \mid 0 \le k < m_a, a \in V(F(\mu))\}$ is a bounded subset of $k(\alpha,\mathbf{E})$. Since $\rho : \mathrm{Exp}(\mathbb{C}) \to k(\alpha,\mathbf{E})$ is a surjective topological homomorphism by 2.5(6), there exists a bounded set M in $\mathrm{Exp}(\mathbb{C})$ with $\rho(M) = B$ and consequently:

(4) There exists $F > 0$ such that for each $a \in V(F(\mu))$ and each $0 \le k < m_a$ one can find $f_{k,a} \in \mathrm{Exp}(\mathbb{C})$ with $\rho(f_{k,a}) = y_{k,a}$ and

$$\sup_{a\in V(F(\mu))} \sup_{0\le k<m_a} \sup_{z\in\mathbb{C}} |f_{k,a}(z)| e^{-F|z|} < \infty.$$

Now we have proved the following proposition:

3.7 PROPOSITION. *Let* T_μ *denote a non-zero convolution operator on* $A(\mathbb{C})$ *for which* $\ker T_\mu$ *is infinite dimensional and which has the following property:*

$$* \begin{cases} \text{There exist positive numbers } \varepsilon, C \text{ and } E \text{ such that, for every component } S \text{ of} \\ \{z \in \mathbb{C} \mid |F(\mu)[z]| < \varepsilon e^{C|z|}\} \text{ with } S \cap V(F(\mu)) \neq \emptyset, \text{ we have} \\ (\operatorname{diam} S)^{m_S} \exp(-E \sup_{z\in S} |z|) \le E, \text{ where } m_S := \max\{m_a - 1 \mid a \in S \cap V(F(\mu))\}. \end{cases}$$

Then, if the exponential monomials $\{z^k e^{az} \mid 0 \le k < m_a, a \in V(F(\mu))\}$ *form a Schauder basis of* $\ker T_\mu$, *assertion 3.6(4) holds.*

Remark. a) From the estimates noted in 2.5(1), it follows easily that the hypothesis (*) in 3.7 is satisfied whenever $\sup\{m_a \mid a \in V(F(\mu))\} < \infty$.
b) It is possible to show that 3.6(4) is equivalent to the fact that the multiplicity variety of the ideal $F(\mu)\mathrm{Exp}(\mathbb{C})$ is an interpolating variety in the notation of Berenstein and Taylor [1]. In view of this, it follows from Berenstein and Taylor [1], thm. 4, and some additional considerations that, under the hypotheses of 3.7, the following assertions are equivalent:

(i) The exponential monomials form a Schauder basis of $\ker T_\mu$;
(ii) 3.6(4) holds;
(iii) there exists $A > 0$ with $\inf_{a\in V(F(\mu))} \left| \frac{F(\mu)^{(m_a)}(a)}{m_a!} \right| e^{A|a|} > 0.$

The necessary condition given in 3.7 looks rather complicated; however, it is easy to derive the following simple explicit condition from it.

3.8 COROLLARY. *Let* T_μ *denote a convolution operator on* $A(\mathbb{C})$ *which satisfies the hypotheses of Proposition 3.7. If the exponential monomials* $\{z^k e^{az} \mid 0 \le k < m_a, a \in V(F(\mu))\}$ *form a Schauder basis of* $\ker T_\mu$, *then there exists* $F > 0$ *such that* $\inf_{a\in V(F(\mu))} \operatorname{dist}(a, V(F(\mu))\setminus\{a\}) e^{F|a|} > 0$.

PROOF. From 3.7 we get a positive number F such that, for each $a \in V(F(\mu))$, there exists $f_a \in \mathrm{Exp}(\mathbb{C})$ with the properties $\sup_{a\in V(F(\mu))} \sup_{z\in\mathbb{C}} |f_a(z)| e^{-F|z|} \le A < \infty$, $f_a(a) = 1$

and $f_a(b) = 0$ for all $b \in V(F(\mu)) \setminus \{a\}$. Now fix $a \in V(F(\mu))$ and choose $b \in V(F(\mu))$ with $|b-a| = \text{dist}(a, V(F(\mu)) \setminus \{a\})$. Without loss of generality, we can assume $|b-a| < 1$. Letting $\mathbb{D}$ denote the unit disk, we define $g : \mathbb{D} \to \mathbb{C}$ by $g(w) := f_b(a+w)$. Then we have $g(0) = f_b(a) = 0$ and $\|g\|_{A^\infty(\mathbb{D})} \leq A \sup_{|w| \leq 1} e^{F|a+w|} = Ae^F e^{F|a|}$. Now the Schwarz lemma implies $|g(w)| \leq |w| Ae^F e^{F|a|}$, and hence

$$1 = f_b(b) = g(b-a) \leq |b-a| Ae^F e^{F|a|},$$

whence the desired condition.

Finally we show how Corollary 3.8 can be used to construct examples of convolution operators T_μ for which the exponential monomials do not form a Schauder basis. This is done by jiggling the zeros of certain functions (see Berenstein and Taylor [1], p. 120).

3.9 EXAMPLE. Let $f \in \text{Exp}(\mathbb{C})$ be a function for which $V(f)$ is infinite and for which $m_a = 2$ for all $a \in V(f)$, e.g. $f(z) = \left(\sum_{k=0}^{\infty} \frac{z^k}{(2k)!}\right)^2$. Label the elements of $V(f)$ by $(a_k)_{k \in \mathbb{N}}$ in such a way that $(|a_k|)_{k \in \mathbb{N}}$ is non-decreasing. Next, choose a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ of complex numbers with the following properties:

(1) $(\varepsilon_k)_{k \in \mathbb{N}} \in \Lambda^1_\infty(|a_k|)$,

(2) $|\varepsilon_k| > 0$ for all $k \in \mathbb{N}$,

(3) $\sum_{k=1}^{\infty} |\varepsilon_k| < 1$,

(4) $V(f) \cap (a_k + \varepsilon_k \mathbb{D}) = \{a_k\}$

and put $g : z \mapsto f(z) \prod_{k=1}^{\infty} \frac{z-(a_k+\varepsilon_k^2)}{z-a_k}$ for $z \in \mathbb{C} \setminus V(f)$. Since $m_{a_k} = 2$ for all $k \in \mathbb{N}$, g defines an entire function and, in view of our choice, it is not difficult to show that $g \in \text{Exp}(\mathbb{C})$. Obviously, $V(g) = V(f) \cup \{a_k + \varepsilon_k^2 \mid k \in \mathbb{N}\}$, and every zero of g is simple. Hence it follows from (1) and Corollary 3.8 that the exponential monomials do not form a Schauder basis in $\ker T_\mu$ if we put $\mu := F^{-1}(g)$.

By Example 3.9, it is clear that, as a general answer to question 1.3(a), Theorem 3.4 is optimal.

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SOME RESULTS ON CONTINUOUS LINEAR MAPS BETWEEN FRECHET SPACES

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The present article is mainly concerned with the derived functors $Ext^1(E,\cdot)$ of the functors $L(E,\cdot)$, E fixed, acting from the category of Fréchet spaces to the category of linear spaces (see Palamodov [17]), with conditions for $Ext^1(E,F) = 0$ (see [31]) and their applications. $Ext^1(E,F) = 0$ means that all exact sequences $0 \to F \to G \to E \to 0$ split (cf. Thm. 1.8.).

In a first section we give a short introduction to the theory of the functors $Ext^1(E,\cdot)$ (cf. [17], [31]) with special emphasis on the concrete representations for $Ext^1(E,F)$ and the connecting maps in case E or F is nuclear. The necessary resp. sufficient conditions for $Ext^1(E,F) = 0$ (Thm. 1.9.) are presented without proof (s. [31]). Instead we give in §2 a direct proof for $Ext^1(s,s) = 0$ which leads via the permanence properties derived in § 1 directly to the splitting theorem for exact sequences $0 \to F \to G \to E \to 0$ where F is a quotient and E a subspace of s (cf. [23], [34], [24]). The description of the classes $Ext^1(E,s) = 0$ and $Ext^1(s,F) = 0$ given in Thm. 2.5. shows the strong connection between the theory of $Ext^1(\cdot,\cdot)$ and the structure theory of nuclear Fréchet spaces as developed for the case of s in [23] and [34].

In § 3 we prove by use of the properties of $Ext^1(\omega,\cdot)$ and a lemma from [29] (Lemma 2.1. in the present paper) that a hypoelliptic partial differential operator on $\mathbb{R}^n$ with constant coefficients has no right inverse in $C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open (see [32] with a different proof). This extends a result of Grothendieck on elliptic operators.

In § 4 we show how the knowledge of conditions for $Ext^1(E,F) = 0$ can be used for the investigation of topological properties (barrelledness etc.) of spaces $E_b' \hat{\otimes}_\pi F \cong L_b(E,F)$, E,F Fréchet spaces, one of them nuclear (cf. Grothendieck [8], II, § 4). In particular we give in Thm. 4.9. a complete basis free description of the classes $\{E ; L_b(E,F_0)$ barrelled$\}$ and $\{F : L_b(E_0,F)$ barrelled$\}$ if E_0 or F_0 is a power series space satisfying a stability condition, so extending the results of Grothendieck loc. cit.. We also give a solution for the problem of classification of complex manifolds V according to topological properties (here barrelledness of $L_b(E,F)$) as proposed in Grothendieck [8], II, p. 128, at least if V is a Stein manifold. A more direct and systematic treatment is contained in [33].

In § 4 we make use of the results of [31], § 4 and § 5 on $\mathrm{Ext}^1(E,F) = 0$ which are based on (S_1^*) and (S_2^*). The proofs are rather complicated so we do not present them here. Instead we use in the final section conditions (S_1^*) and (S_2^*) to investigate which spaces E or F can occur in a nontrivial way in the relation $\mathrm{Ext}^1(E,F)=0$. These are essentially the countably normed spaces E (more precise: the spaces E satisfying a condition (DN_φ)) and the quasinormable spaces F. A special role play the spaces F which are quojections or do not satisfy the condition (*) of Bellenot-Dubinsky [3]. Some results of § 5 are strongly related to the results of [14].

0. We will use standard notation of the theory of locally convex spaces as in [12], [20]. For nuclear spaces we refer to [8], [19], for sequence spaces to [5] and for concepts of homological algebra to [17] and [15].

Throughout the paper E,F,G,H always denote locally convex Frêchet spaces, over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\| \ \|_1 \le \| \ \|_2 \le \dots$ a fundamental system of seminorms. L(E,F) is the linear space of continuous linear maps from E to F. On the dual space $E'=L(E,\mathbb{K})$ of E we consider the dual ($\mathbb{R}_+ \cup \{+\infty\}$-valued) norms

$$\|y\|_k^* = \sup\{|y(x)| : x \in E, \|x\|_k \le 1\}.$$

We put

$$E_k' := \{y \in E' : \|y\|_k^* < +\infty\}.$$

A Frêchet space E is said to have property (DN) (resp. (Ω)) if there exists a fundamental system of seminorms such that $\| \ \|_k^2 \le \| \ \|_{k-1} \| \ \|_{k+1}$ (resp. $\| \ \|_k^{*2} \le \| \ \|_{k-1}^* \| \ \|_{k+1}^*$) for all k (see [23], [34], [30]).

A nuclear Frêchet space is isomorphic to a subspace of s iff it has property (DN) (s. [23]), it is isomorphic to a quotient space of s iff it has property (Ω), (s. [34]). s denotes the nuclear Frêchet space of all rapidly decreasing sequences:

$$s = \{x = (x_1,x_2, \dots) : \|x\|_k = \sum_j |x_j| \, j^k < +\infty \quad \text{for all } k\}.$$

More generally let $A = (a_{j,k})_{j,k}$ be an infinite matrix such that $0 \le a_{j,k} \le a_{j,k+1}$, $\sup_k a_{j,k} > 0$ for all j and k. Then we define

$$\lambda(A) := \{x = (x_1,x_2, \dots) : \|x\|_k = \sum_j |x_j| \, a_{j,k} < +\infty \text{ for all } k\}$$

$$\lambda^\infty(A) := \{x = (x_1,x_2, \dots) : \|x\|_k = \sup_j |x_j| \, a_{j,k} < +\infty \text{ for all } k\}.$$

Equipped with their respective seminorms $(\| \ \|_k)_{k \in \mathbb{N}}$ these are Frêchet spaces. They are called Köthe sequence spaces. They are nuclear iff for every k there is p such that (with $\frac{0}{0} = 0$) :

$$\sum_j \frac{a_{j,k}}{a_{j,k+p}} < +\infty \qquad \text{(Grothendieck-Pietsch criterion).}$$

In this case we have $\lambda(A) = \lambda^\infty(A)$, and vice versa.

If $a_{j,k} = \rho_k^{\alpha_j}$ with sequences $0 < \rho_1 < \rho_2 < \ldots \nearrow r$ and $\alpha = (\alpha_j)_j : \alpha_1 \leq \alpha_2 \leq \ldots \nearrow +\infty$, then $\lambda(A)$ (resp. $\lambda^\infty(A)$) is called power series space and denoted by $\Lambda_r(\alpha)$ (resp. $\Lambda_r^\infty(\alpha)$). It depends only on r and the sequence α. For fixed α all spaces $\Lambda_r(\alpha)$ (resp. $\Lambda_r^\infty(\alpha)$) with $r < +\infty$ are isomorphic. Therefore we can restrict our attention to the cases $r = 1, +\infty$. $\Lambda_1(\alpha)$ is called power series space of finite type, $\Lambda_\infty(\alpha)$ of infinite type. Power series spaces of finite type and of infinite type can never be isomorphic.

By $\omega := \mathbb{K}^{\mathbb{N}}$ we denote the product of countably many copies of the scalar field, by $\mathcal{H}(V)$ the nuclear Frêchet space of holomorphic functions on a complex manifold V.

1. In this first section we present some basic facts on the derived functors $\mathrm{Ext}^k(E,\cdot), k = 0,1,\ldots$, of the functor $L(E,\cdot)$. All these functors are considered to act from the category of Frêchet spaces to the category of linear spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. E denotes a fixed Frêchet space.

We do not give a construction for these functors but take their existence as granted by homological algebra (see Palamodov [17]). Instead we take an axiomatic approach, i.e. we state properties of these functors and this will be the only information which we will use.

The functors $\mathrm{Ext}^k(E,\cdot)$ have the following properties:

(0) $\mathrm{Ext}^0(E,\cdot) = L(E,\cdot)$

(I) To every short exact sequence $0 \to F \xrightarrow{\iota} G \xrightarrow{q} H \to 0$ there are assigned linear maps

$$\delta^k : \mathrm{Ext}^k(E,H) \to \mathrm{Ext}^{k+1}(E,F),\ k = 0,1,\ \ldots$$

such that

$$0 \to L(E,F) \xrightarrow{\iota^*} L(E,G) \xrightarrow{q^*} L(E,H) \xrightarrow{\delta^0} \mathrm{Ext}^1(E,F) \xrightarrow{\iota^1} \mathrm{Ext}^1(E,G) \xrightarrow{q^1} \mathrm{Ext}^1(E,H)$$
$$\xrightarrow{\delta^1} \mathrm{Ext}^2(E,F) \xrightarrow{\iota^2} \ldots$$

is exact and depends functorially on the short exact sequence.

(II) For every injective space I we have $\mathrm{Ext}^k(E,I) = 0$ for $k = 1,2,\ \ldots$.

We used (and will use) the following notations: $\varphi \circ A = \varphi^* A$, $\varphi^k A$ is the map assigned to A by the functors $\mathrm{Ext}^k(E,\cdot)$, $k = 1,2, \dots$. A space I is called injective if for each space E, each closed subspace $E_0 \subset E$, and each $\varphi \in L(E_0,I)$ there exists an extension $\Phi \in L(E,I)$. Examples of injective spaces are the spaces $l^\infty(M)$, M an index set, and their products.

If F is a Fréchet space then an exact sequence

$$0 \to F \to I_0 \xrightarrow{\iota_0} I_1 \xrightarrow{\iota_1} I_2 \to \dots ,$$

I_k injective for all k, is called an injective resolution.

We can use any injective resolution of F to calculate the space $\mathrm{Ext}^k(E,F)$, as the following proposition shows. The proof is standard. In fact usually one uses injective resolutions to prove the existence of functors $\mathrm{Ext}^k(E,\cdot)$ with the properties described above.

1.1. Proposition: *If* $0 \to F \to I_0 \xrightarrow{\iota_0} I_1 \xrightarrow{\iota_1} I_2 \to \dots$ *is an injective resolution, then*

$$\mathrm{Ext}^k(E,F) \cong \ker \iota_k^* \,/\, \mathrm{im}\, \iota_{k-1}^*$$

for $k = 1,2, \dots$.

Proof: We put $Z_0 = F$, $Z_k = \ker \iota_k = \mathrm{im}\, \iota_{k-1}$ for $k = 1,2, \dots$ and obtain for each $k = 0,1, \dots$ a short exact sequence

$$0 \to Z_k \longrightarrow I_k \xrightarrow{\iota_k} Z_{k+1} \to 0$$

which leads by (I) and (III) to a long exact sequence

$$0 \to L(E,Z_k) \to L(E,I_k) \xrightarrow{\iota_k^*} L(E,Z_{k+1}) \to \mathrm{Ext}^1(E,Z_k) \to 0 \to \mathrm{Ext}^1(E,Z_{k+1}) \to$$
$$\to \mathrm{Ext}^2(E,Z_k) \to 0 \to \dots .$$

Since $L(E,Z_{k+1}) = \ker \iota_{k+1}^*$ this gives us for all $k = 0,1,2, \dots$, $j = 1,2, \dots$:

$$\mathrm{Ext}^1(E,Z_k) \cong \ker \iota_{k+1}^* \,/\, \mathrm{im}\, \iota_k^*$$

$$\mathrm{Ext}^j(E,Z_{k+1}) \cong \mathrm{Ext}^{j+1}(E,Z_k) .$$

Combining these equations and using $Z_0 = F$ we obtain

$$\mathrm{Ext}^k(E,F) \cong \mathrm{Ext}^1(E,Z_{k-1}) \cong \ker \iota_k^* \,/\, \mathrm{im}\, \iota_{k-1}^*$$

for $k = 1,2, \dots$.

It can easily be seen that every Banach space F has an injective resolution of Banach spaces. We need only to know that every Banach space X can be naturally imbedded into $l^\infty(B^0)$ where B^0 is the unit ball in X'. We apply this to F and put

$I_o = l^\infty(B^o)$, then to I_o/F and obtain I_1, etc. .

1.2. Corollary: *If* E *is nuclear and* $F = \prod_j F_j$ *is a product of Banach spaces* F_j, *then* $\mathrm{Ext}^k(E,F) = 0$ *for* $k = 1,2, \ldots$.

Proof: We apply the above described procedure separately to the F_j and obtain an injective resolution of the form

$$0 \to \prod_j F_j \to \prod_j I_o^j \xrightarrow{\prod_j \iota_1^j} \prod_j I_1^j \xrightarrow{\prod_j \iota_2^j} \ldots$$

where the I_k^j are Banach spaces.

From the properties of a nuclear space (s. Grothendieck [8],II,§ 3, n°1) and Prop. 1.1. we conclude the assertion.

The preceding remark is very useful as one sees from the following lemma which is contained in [17] Thm. 5.2 (or Cor. 5.1.) and a proof of which can be found in [31] (Lemma 1.1.). We need the following notation:

A projective spectrum $\rho_{l,k} : F_l \to F_k$ $(l \geq k)$ of Banach spaces is called a fundamental system of Banach spaces for the Fréchet space F if
(i) $F = \lim \operatorname{proj} F_k$
(ii) for every k there is an $l \geq k$ such that $\rho_k F$ is dense in $\rho_{l,k} F_l$, where $\rho_k : F \to F_k$ denotes the canonical map.

1.3. Lemma: *If* $\rho_{l,k} : F_l \to F_k$ *is a fundamental system of Banach spaces for* F *then the sequence ("canonical resolution")*

$$0 \to F \xrightarrow{\iota} \prod_k F_k \xrightarrow{q} \prod_k F_k \to 0$$

is exact, where $\iota : x \to (\rho_k x)_k$ *and* $q : (x_k)_k \to (\rho_{k+1,k}\, x_{k+1} - x_k)_k$.

This yields as an immediate consequence (see [31], Thm. 1.2.; cf. [16], 7.1., [17] , p. 51) :

1.4. Theorem: *If* E *or* F *is nuclear then* $\mathrm{Ext}^k(E,F) = 0$ *for* $k \geq 2$.

Proof: If F is nuclear, then F has a fundamental system of Banach spaces isomorphic to l^∞. Hence the F_k in Lemma 1.3. can be chosen injective, which means that the short exact sequence in Lemma 1.3. is an injective resolution of length 1. The assertion follows from Prop. 1.1.

If E is nuclear then we choose any fundamental system of Banach spaces and apply (I) to the canonical resolution (see 1.3.). We obtain

$$\ldots \to 0 \to 0 \to \mathrm{Ext}^k(E,F) \to 0 \to 0 \ldots$$

for $k = 2,3, \ldots$. The zeros come from Cor. 1.2. .

From Thm. 1.4. we obtain a permanence property:

1.5. Corollary: *If* E *or* F *is nuclear,* $\mathrm{Ext}^1(E,F) = 0$ *and* F_0 *a quotient of* F, *then also* $\mathrm{Ext}^1(E,F_0) = 0$.

Proof: Let $q : F \to F_0$ be the quotient map, $G := \ker q$. Then (I) applied to the short exact sequence

$$0 \to G \to F \to F_0 \to 0$$

gives

$$\ldots \to \mathrm{Ext}^1(E,F) \to \mathrm{Ext}^1(E,F_0) \to \mathrm{Ext}^2(E,G) \to \ldots .$$

The first term is zero by assumption, the third by 1.4., hence also the middle term.

Our next task is to get more information on $\mathrm{Ext}^1(E,F)$ in case one of the spaces is nuclear.

1.6. Theorem: *Let* $\mathcal{F} = \{F_k, \rho_{l,k}\}$ *be a fundamental system of Banach spaces for* F, *let* E *or* F *be nuclear. Then*

$$\mathrm{Ext}^1(E,F) \cong \prod_k L(E,F_k) \,/\, B(E,\mathcal{F})$$

where

$$B(E,\mathcal{F}) = \{(A_k)_k \in \prod_k L(E,F_k) : \exists (B_k)_k \in \prod_k L(E,F_k) \text{ such that } A_k = \rho_{k+1,k} B_{k+1} - B_k \text{ for all } k\}.$$

Proof: If E is nuclear we apply (I) to the canonical resolution (see 1.3.). We obtain

$$\ldots \to \prod_k L(E,F_k) \xrightarrow{q^*} \prod_k L(E,F_k) \xrightarrow{\delta^0} \mathrm{Ext}^1(E,F) \to 0 \to \ldots$$

where the zero comes from Cor. 1.2. .

If F is nuclear and $\rho_{l,k} : F_l \to F_k$ is any fundamental system of Banach spaces then we can find a fundamental system $\rho^0_{l,k} : F^0_l \to F^0_k$ of injective Banach spaces and maps $\varphi_k : F^0_k \to F_k$ such that

$$\begin{array}{ccccc} & \overset{\rho_1}{\nearrow} & F_1 & \xrightarrow{\rho_{1,k}} & F_k \\ F & & \uparrow{\scriptstyle \varphi_1} & & \uparrow{\scriptstyle \varphi_k} \\ & \underset{\rho_1^0}{\searrow} & F_1^0 & \xrightarrow[\rho_{1,k}^0]{} & F_k^0 \end{array}$$

is commutative.

Hence we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & F & \to & \prod_k F_k & \xrightarrow{q} & \prod_k F_k & \to & 0 \\ & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \varphi} & & \\ 0 & \to & F & \to & \prod_k F_k^0 & \xrightarrow{q^0} & \prod_k F_k^0 & \to & 0 \end{array}$$

where φ is induced by the φ_k. We apply (I) to this diagram and obtain

$$\begin{array}{ccccccccc} \ldots & \to & \prod_k L(E,F_k) & \xrightarrow{q^*} & \prod_k L(E,F_k) & \xrightarrow{\delta^0} & Ext^1(E,F) & \to & \ldots \\ & & \uparrow{\scriptstyle \varphi^*} & & \uparrow{\scriptstyle \varphi^*} & & \uparrow{\scriptstyle id} & & \\ \ldots & \to & \prod_k L(E,F_k^0) & \xrightarrow{q^{o*}} & \prod_k L(E,F_k^0) & \xrightarrow{\delta^0} & Ext^1(E,F) & \to & 0 \end{array}$$

The zero comes from (II). Obviously the assertion follows from this diagram.

From Thm. 1.6. again we obtain a permanence property:

<u>1.7. Corollary</u>: *If* E *or* F *is nuclear,* $Ext^1(E,F) = 0$ *and* E_0 *a closed subspace of* E, *then* $Ext^1(E_0,F) = 0$.

<u>*Proof:*</u> We choose a fundamental system of Banach spaces for F which for nuclear F we assume to consist of injective Banach spaces. Hence in both cases the restriction maps $R_k : L(E,F_k) \to L(E_0,F_k)$ are surjective for all k (for E nuclear see [8], II, §3). The R_k induce the map R in the following diagram. R is surjective.

$$0 = Ext^1(E,F) \cong \prod_k L(E,F_k) \;/\; B(E,\mathcal{F}) \xrightarrow{R} \prod_k L(E_0,F_k) \;/\; B(E_0,\mathcal{F}) \cong Ext^1(E_0,F)$$

This shows the assertion.

In the following part of this section we want to explain, what in terms of the concrete representation of 1.6. (and 1.4.) $Ext^1(E,F)$ resp. the long exact sequence (I) "means". Let us remark that the isomorphism which we used in 1.6. is in a canonical way determined by the fundamental system of Banach spaces. It is induced by the δ^0 assigned to the canonical resolution (see 1.3.). In the following discussion we will always talk about this isomorphism.

Let $0 \to F \xrightarrow{\iota} G \xrightarrow{q} H \to 0$ be an exact sequence. We assume that either E or F,G and H are nuclear. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be fundamental systems of Banach spaces such that for every k we have an exact sequence of Banach spaces

$$0 \to F_k \xrightarrow{\iota_k} G_k \xrightarrow{q_k} H_k \to 0$$

and such that for all $l > k$ the diagram

$$\begin{array}{ccccccccc}
0 & \to & F & \xrightarrow{\iota} & G & \xrightarrow{q} & H & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_l & \xrightarrow{\iota_l} & G_l & \xrightarrow{q_l} & H_l & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_k & \xrightarrow{\iota_k} & G_k & \xrightarrow{q_k} & H_k & \to & 0
\end{array}$$

is commutative. We can obtain such fundamental systems e.g. by taking for $\mathcal{G}$ the Banach spaces generated by a fundamental system of seminorms on G and for $\mathcal{F}$ and $\mathcal{H}$ the Banach spaces generated by the induced resp. coinduced fundamental systems of seminorms on F resp. H.

We have the following situation

$$\begin{array}{ccccccc}
 & \overset{\delta^0}{\nearrow} \mathrm{Ext}^1(E,F) & \xrightarrow{\iota^1} & \mathrm{Ext}^1(E,G) & \xrightarrow{q^1} & \mathrm{Ext}^1(E,H) & \longrightarrow 0 \\
L(E,H) & \wr\| & & \wr\| & & \wr\| & \\
 & \underset{D}{\searrow} \prod_k L(E,F_k)/B(E,\mathcal{F}) & \xrightarrow{I} & \prod_k L(E,G_k)/B(E,\mathcal{G}) & \xrightarrow{Q} & \prod_k L(E,H_k)/B(E,\mathcal{H}) & \to 0
\end{array}$$

where D,I,Q denote the maps induced by means of the isomorphisms from δ^0, ι^1, q^1 . We want to describe these maps.

We apply therefore (I) columnwise to the following diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
F & \xrightarrow{\iota} & G & \xrightarrow{q} & H \\
\downarrow & & \downarrow & & \downarrow \\
\prod_k F_k & \xrightarrow{\prod_k \iota_k} & \prod_k G_k & \xrightarrow{\prod_k q_k} & \prod_k H_k \\
\downarrow & & \downarrow & & \downarrow \\
\prod_k F_k & \xrightarrow{\prod_k \iota_k} & \prod_k G_k & \xrightarrow{\prod_k q_k} & \prod_k H_k \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where the columns are canonical resolutions. We obtain by use of (I) and (the proof of) 1.6. the following commutative diagram with exact columns. The rows are easily seen to be exact.

$$\begin{array}{ccccc}
0 & & 0 & & 0\\
\downarrow & & \downarrow & & \downarrow\\
L(E,F) & \xrightarrow{\iota^*} & L(E,G) & \xrightarrow{q^*} & L(E,H)\\
\downarrow & & \downarrow & & \downarrow a\\
\prod_k L(E,F_k) & \xrightarrow{\prod_k \iota_k^*} & \prod_k L(E,G_k) & \xrightarrow{\prod_k q_k^*} & \prod_k L(E,H_k)\\
\downarrow & & \downarrow b & & \downarrow c\\
\prod_k L(E,F_k) & \xrightarrow{\prod_k \iota_k^*} & \prod_k L(E,G_k) & \xrightarrow{\prod_k q_k^*} & \prod_k L(E,H_k)\\
\downarrow \delta^0 & & \downarrow \delta^0 & & \downarrow \delta^0\\
\mathrm{Ext}^1(E,F) & \xrightarrow{\iota^1} & \mathrm{Ext}^1(E,G) & \xrightarrow{q^1} & \mathrm{Ext}^1(E,H)\\
\downarrow & & \downarrow & & \downarrow\\
0 & & 0 & & 0
\end{array}$$

We see immediately that the maps I and Q are the natural maps between the respective spaces, i.e. the maps induced by $\prod_k \iota_k$ and $\prod_k q_k$ on the quotients.

Moreover we can by a standard procedure of homological algebra (s. [15]) define a map $d : L(E,H) \to \mathrm{Ext}^1(E,F)$ as follows:

For $\varphi \in L(E,H)$ we have, because of the nuclearity assumptions, $a\varphi \in \operatorname{im} (\prod q_k^*)$. We choose $\psi \in \prod L(E,G_k)$ such that $(\prod q_k^*)\ \psi = a\varphi$. Since $(\prod q_k^*)\ b\ \psi = c(\prod q_k^*)\ \psi = c\ a\varphi = 0$ we can find $\chi \in \prod L(E,F_k)$ such that $(\prod \iota_k^*)\ \chi = b\ \psi$. We put $d\ \varphi := \delta^0\chi$. It is not difficult to prove that d is well defined, linear and makes the following sequence exact

$$\dots \to L(E,G) \xrightarrow{q^*} L(E,H) \xrightarrow{d} \mathrm{Ext}^1(E,F) \xrightarrow{\iota^1} \mathrm{Ext}^1(E,H) \to \dots$$

Hence $\delta^0 = A \circ d$ where A is an automorphism of $\ker \iota^1$ and δ^0 is the one acting from $L(E,H) \to \mathrm{Ext}^1(E,F)$.

We now put $\tilde{D}\varphi := [\chi]$ where [ ] denotes the equivalence class in $\prod_k L(E,F_k)/B(E,\mathcal{F})$. It follows easily ($\delta^0$ acting from $\prod_k L(E,F_k) \to \mathrm{Ext}^1(E,F)$ induces an isomorphism) that also $\tilde{D}$ is well defined and linear. Straightforward calculation shows that $D = B \circ \tilde{D}$ where B is some automorphism of ker I, or equivalently: that $\operatorname{im} D = \operatorname{im} \tilde{D}$, $\ker D = \ker \tilde{D}$.

We have to explain $\tilde{D}$. It describes the following (Mittag-Leffler-) approach to the solution of the lifting problem

$$\begin{array}{ccccccccc} 0 & \to & F & \to & G & \overset{q}{\to} & H & \to & 0 \\ & & & & & \nwarrow & \uparrow \varphi & & \\ & & & & & & E & & \end{array}$$

First we solve all the "local" lifting problems

$$\begin{array}{ccccccccc} 0 & \to & F_k & \to & G_k & \overset{q_k}{\to} & H_k & \to & 0 \\ & & & & & \nwarrow \psi_k & \uparrow \varphi_k & & \\ & & & & & & E & & \end{array}$$

which we can do by our nuclearity assumptions. The q_k are the maps induced by q, i.e. the components of aq. The φ_k give the components of $a\varphi$. Then we apply b, i.e. form the maps $\rho_{k+1,k} \circ \psi_{k+1} - \psi_k$. If we think of ι as an imbedding, these can be considered as maps in $L(E,F_k)$, i.e. they define the components χ_k of an element $\chi \in \prod_k L(E,F_k)$.

Our original lifting problem is easily seen to be solvable if and only if we can find $(B_k)_k \in \prod_k L(E,F_k)$ such that $\rho_{k+1,k} \circ \psi_{k+1} - \psi_k = \rho_{k+1,k} \circ B_{k+1} - B_k$, hence $\rho_{k+1,k}(\psi_{k+1} - B_{k+1}) = \psi_k - B_k$ for all k. This means nothing else than $\tilde{D}\varphi = [\chi] = 0$ and that is equivalent to $D\varphi = 0$.

Hence the axiomatic approach describes (under our nuclearity assumptions) via the concrete representation nothing else than the possibility of the "natural" way of solution of the lifting problem. ker ι^1 describes the obstruction against this in the concrete situation. $\mathrm{Ext}^1(E,F)$ describes the "structural" or "maximal" obstruction in any situation. If it is zero the procedure always works.

We admit without further proof the following theorem (s. [31], Thm 1.8.):

<u>1.8. Theorem</u>: *The following are equivalent:*

(1) $\mathrm{Ext}^1(E,F) = 0$

(2) *Every exact sequence* $0 \to F \to G \to E \to 0$ *splits.*

(3) *For every exact sequence* $0 \to F \to G \overset{q}{\to} H \to 0$ *and* $\varphi \in L(E,H)$ *there exists* $\psi \in L(E,G)$ *with* $\varphi = q \circ \psi$.

(4) *For every exact sequence* $0 \to H \overset{\iota}{\to} G \to E \to 0$ *and* $\varphi \in L(H,F)$ *there exists* $\psi \in L(G,F)$ *with* $\varphi = \psi \circ \iota$.

The rest of this paper will be mainly devoted to a discussion of conditions for $\mathrm{Ext}^1(E,F) = 0$ and their applications. We use the following conditions on two

Fréchet spaces E and F (s. [31] and for (S_1^*) Apiola [2]). $\| \|_1 \le \| \|_2 \le \dots$ denotes a fundamental system of seminorms on E or F resp., $\|y\|_k^* = \sup \{|y(x)| : \|x\|_k \le 1\}$ the dual norm of $\| \|_k$ for $y \in E'$ or F'.

(S_1^*) $\exists n_0\ \forall \mu\ \exists k\ \forall K,m\ \exists n,S\ \forall x \in E,\ y \in F'_\mu$

$$\|x\|_m \|y\|_k^* \le S \left(\|x\|_n \|y\|_K^* + \|x\|_{n_0} \|y\|_\mu^* \right)$$

(S_2^*) $\forall \mu\ \exists n_0,k\ \forall K,m\ \exists n,S\ \forall x \in E,\ y \in F'_\mu$

$$\|x\|_m \|y\|_k^* \le S \left(\|x\|_n \|y\|_K^* + \|x\|_{n_0} \|y\|_\mu^* \right)$$

In [31], Thm. 3.7. in connection with 3.9. it is shown:

1.9. Theorem: *If* E *is countably normable or* F *reflexive and one of them nuclear, then*

$$(S_1^*) \Rightarrow \mathrm{Ext}^1(E,F) = 0 \Rightarrow (S_2^*) .$$

In [13] it is shown that (S_2^*) is necessary and sufficient for $\mathrm{Ext}^1(E,F) = 0$ in the case of Köthe spaces. A necessary and sufficient condition for $\mathrm{Ext}^1(E,F) = 0$ in the general case is contained in [33].

Further results on splitting relations, generalizations and investigations of conditions for an exact sequence to split if $\mathrm{Ext}^1(E,F) \ne 0$ are contained in [1], [1o] , [18], [25], [26].

2. In this section all spaces are assumed to be nuclear. Instead of presenting details of the rather complicated proof of Thm. 1.9. we discuss with full proofs an example which immediately leads to the most important cases of $\mathrm{Ext}^1(E,F) = 0$ with respect to applications. It shows in a very nice way the interplay between the structure theory of (s) as developed in [23], [34] (cf. also [5], [24], [35]) and the present theory. The following lemma 2.1. is in an equivalent form (s. Thm. 1.8.) contained in [23], 1.5.. We give a direct proof using the matrices of the respective maps.

2.1. Lemma: $\mathrm{Ext}^1(s,s) = 0$

Proof: We apply 1.3. or 1.5. to the fundamental system

$$F_k = \{x = (x_1,x_2, \dots) : \|x\|_k = \sup_j |x_j|\, 2^{kj} < +\infty\}$$

of injective Banach spaces. Let $A_k \in L(s,F_k)$, $k = 1,2, \dots$ be given. Each A_k is represented by a matrix $(a_{\nu,j}^{(k)})_{\nu,j \in \mathbb{N}}$ with

$$\sup_{\nu,j} |a^{(k)}_{\nu,j}| \, 2^{kj-n\nu} \le 2^d < +\infty$$

for some $n = n(k)$, $d = d(k)$.

We have to determine matrices $(b^{(k)}_{\nu,j})$ satisfying analogous estimates such that $a^{(k)}_{\nu,j} = b^{(k+1)}_{\nu,j} - b^{(k)}_{\nu,j}$ for all k,ν,j.

We put $D(1) = d(1)$, $D(k+1) = \max\,(d(k+1), 2\,D(k) + k + 3)$ and $N(1) = n(1)$, $N(k+1) = \max\,(n(k+1), 2\,N(k))$ for all k.

We will determine inductively matrices $(u^{(k)}_{\nu,j})$ for $k = 2,3, \ldots$ and $(v^{(k)}_{\nu,j})$ for $k = 1,2, \ldots$ with

(1) $|u^{(k)}_{\nu,j}| \, 2^{(k-2)j} \le 2^{-k}$

(2) $|v^{(k)}_{\nu,j}| \, 2^{kj} \le 2^{D(k)+N(k)\nu}$

(3) $u^{(k+1)}_{\nu,j} + v^{(k+1)}_{\nu,j} = a^{(k)}_{\nu,j} + v^{(k)}_{\nu,j}$

for all k,ν,j.

We start with $v^{(1)}_{\nu,j} = 0$ for all ν,j. Let $v^{(k)}_{\nu,j}$ be determined. We define

$$I_0 = \{(\nu,j) : D(k) + N(k)\nu + k + 2 < j\}$$

$$I_1 = \{(\nu,j) : D(k) + N(k)\nu + k + 2 \ge j\}$$

$$u^{(k+1)}_{\nu,j} = \begin{cases} a^{(k)}_{\nu,j} + v^{(k)}_{\nu,j} & \text{for } (\nu,j) \in I_0 \\ 0 & \text{otherwise} \end{cases}$$

$$v^{(k+1)}_{\nu,j} = \begin{cases} a^{(k)}_{\nu,j} + v^{(k)}_{\nu,j} & \text{for } (\nu,j) \in I_1 \\ 0 & \text{otherwise} \end{cases}$$

Hence (3) is fulfilled by definition.

For $(\nu,j) \in I_0$ we obtain

$$\begin{aligned} |u^{(k+1)}_{\nu,j}| \, 2^{(k-1)j} &\le (|a^{(k)}_{\nu,j}| + |v^{(k)}_{\nu,j}|) \; 2^{(k-1)j} \\ &\le 2^{d(k)+n(k)\nu-j} + 2^{D(k)+N(k)\nu-j} \\ &\le 2^{1+D(k)+N(k)\nu-j} \\ &\le 2^{-k-1} \end{aligned}$$

and for $(\nu,j) \in I_1$

$$|v^{(k+1)}_{\nu,j}| \, 2^{(k+1)j} \leq (|a^{(k)}_{\nu,j}| + |v^{(k)}_{\nu,j}|) \, 2^{(k+1)j}$$
$$\leq 2^{1+D(k)+N(k)\nu+j}$$
$$\leq 2^{2D(k)+2N(k)\nu+k+3}$$
$$\leq 2^{D(k+1)+N(k+1)\nu}$$

which proves (1) and (2).

Now we define

$$b^{(k)}_{\nu,j} := v^{(k)}_{\nu,j} - \sum_{l=k+1}^{\infty} u^{(l)}_{\nu,j} = v^{(k+1)}_{\nu,j} - a^{(k)}_{\nu,j} - \sum_{l=k+2}^{\infty} u^{(l)}_{\nu,j} \quad .$$

The second equality comes from (3).

The series converges because of (1). We obtain the following estimate:

$$|b^{(k)}_{\nu,j}| \, 2^{kj} \leq 2^{D(k+1)+N(k+1)\nu} + 2^{d(k)+n(k)\nu} + \sum_{l=k+2}^{\infty} 2^{-l}$$
$$\leq C \quad 2^{N(k+1)\nu}$$

for appropriate $C = C(k)$ and all k,ν,j.

Hence $(b^{(k)}_{\nu,j})_{\nu,j \in \mathbb{N}}$ defines a map $B_k \in L(s,F_k)$.

We further have because of (3)

$$b^{(k+1)}_{\nu,j} - b^{(k)}_{\nu,j} = v^{(k+1)}_{\nu,j} - v^{(k)}_{\nu,j} + u^{(k+1)}_{\nu,j} = a^{(k)}_{\nu,j}$$

which means that $\rho_{k+1,k} \circ B_{k+1} - B_k = A_k$.

From 1.5. and 1.7. we get immediately

2.2. Theorem: *If* E *is a subspace of* s *and* F *a quotient space of* s *then* $\mathrm{Ext}^1(E,F) = 0$.

For the following result we use the method of [23], Satz 1.7., it is indicated in the remark at the end of proof of Satz 1.7. .

2.3. Lemma: *If* $\mathrm{Ext}^1(E,s) = 0$ *then* E *is isomorphic to a subspace of* s.

Proof: There exists an exact sequence ([23] , 1.6.)

$$0 \to s \to s \overset{q}{\to} s^{\mathbb{N}} \to 0$$

According to T. and Y. Komura's theorem ([11]) we can assume E imbedded in $s^{\mathbb{N}}$. We set $\tilde{E} = q^{-1} E$ and obtain an exact sequence

$$0 \to s \to \tilde{E} \to E \to 0$$

which splits by assumption (and Thm. 1.8.). Hence we have an imbedding $E \to \tilde{E} \to s$.

The $\mathrm{Ext}^1(s,\cdot)$ case is a little bit more difficult. The proof uses the method first used in [34] in the slightly changed form of [24].

<u>2.4. Lemma</u>: *If* $\mathrm{Ext}^1(s,F) = 0$ *then* F *is isomorphic to a quotient space of* s.

Proof: We consider F as imbedded in $s^{\mathbb{N}}$, call Q the quotient space and obtain $\tilde{Q}$ as $\tilde{E}$ in the previous proof. We get the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & E & \to & s^{\mathbb{N}} & \xrightarrow{q_1} & Q & \to & 0 \\
 & & & & \uparrow P_1 & & \uparrow q_2 & & \\
0 & \to & E & \to & H & \xrightarrow{P_2} & \tilde{Q} & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & s & & s & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $H = \{(x,y) \in s^{\mathbb{N}} \times \tilde{Q} : q_1 x = q_2 y\}$, $P_1(x,y) = x$, $P_2(x,y) = y$.

Since $\tilde{Q} \subset s$ we get from $\mathrm{Ext}^1(s,E) = 0$ and 1.7. that $\mathrm{Ext}^1(\tilde{Q},E) = 0$. Therefore the second row splits which gives $H = E \oplus \tilde{Q}$.

The first column of the diagram above gives the first row of the diagram below. Our standard exact sequence gives the right column. The rest is constructed along the same line as above .

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & s & \to & H & \to & s^{\mathbb{N}} & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & s & \to & G & \to & s & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & s & & s & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Because of 2.1. the second row splits. Hence $H \cong E \oplus \tilde{Q}$ is a quotient of $G \cong s \oplus s \cong s$ and therefore also E.

Now we have all ingredients for the following theorem which describes the exact solution classes of $\text{Ext}^1(E,s) = 0$, $\text{Ext}^1(s,E) = 0$. Remember that all spaces in this section are assumed to be nuclear.

2.5. Theorem: *(a)* $\text{Ext}^1(E,s) = 0$ *if and only if* E *is isomorphic to a subspace of* s. *(b)* $\text{Ext}^1(s,E) = 0$ *if and only if* E *is isomorphic to a quotient space of* s.

The classes of subspaces and quotient spaces of s have been described in [23], [34] by topological linear invariants (DN) and (Ω). It is interesting with respect to this to compare Thm. 2.5. with [31], § 4 where we describe e.g. the classes $\{E : \text{Ext}^1(E,s) = 0\}$ and $\{F : \text{Ext}^1(s,F) = 0\}$ by topological invariants.

3. In this section we use the theory of § 1 to show that hypoelliptic partial differential operators with constant coefficients have no right inverses in $C^\infty(\Omega)$, where Ω is an open set in $\mathbb{R}^n$. This extends a result of Grothendieck on elliptic operators (s. [21], [22]). It is shown by a different proof and in greater generality in [32].

Let P(D) be a hypoelliptic linear partial differential operator with constant coefficients, $\Omega \subset \mathbb{R}^n$ open and P(D)-convex. We put

$$\mathcal{N}(\Omega) = \{f \in C^\infty(\Omega) : P(D)f = 0\}.$$

From the exact sequence

$$0 \to \mathcal{N}(\Omega) \overset{\iota}{\to} C^\infty(\Omega) \xrightarrow{P(D)} C^\infty(\Omega) \to 0$$

we obtain by (I) for any nuclear Frêchet space E an exact sequence

$$0 \to L(E,\mathcal{N}(\Omega)) \xrightarrow{\iota^*} L(E,C^\infty(\Omega)) \xrightarrow{P(D)^*} L(E,C^\infty(\Omega)) \xrightarrow{\delta^0} \text{Ext}^1(E,\mathcal{N}(\Omega)) \xrightarrow{\iota^1} \text{Ext}^1(E,C^\infty(\Omega)) \to \ldots$$

From [29], 2.1. we take the following lemma. It is shown there under the assumption of ellipticity but the proof needs only hypoellipticity. For the sake of completeness we give a proof.

3.1. Lemma: $\iota^1 = 0$.

Proof: We choose a sequence $\Omega_1 \subset\subset \Omega_2 \subset\subset \ldots \subset\subset \Omega$ of open sets such that $\Omega = \bigcup_k \Omega_k$ and denote by H_k the closure of $\mathcal{N}(\Omega)_{|\overline{\Omega}_k}$ in $C(\overline{\Omega}_k)$ considered as a space of functions on Ω_k. We obtain in a natural way a commutative diagram with exact lines:

$$\begin{array}{ccccccccc} 0 & \to & C^\infty(\Omega) & \to & \prod_k C^\infty(\Omega_k) & \overset{q}{\to} & \prod_k C^\infty(\Omega_k) & \to & 0 \\ & & \uparrow\iota & & \uparrow j & & \uparrow j & & \\ 0 & \to & \mathcal{N}(\Omega) & \to & \prod_k H_k & \overset{q}{\to} & \prod_k H_k & \to & 0 \end{array} .$$

ι and j are the identical imbeddings, q is defined by $q((f_k)_k) = (f_{k+1}|_{\Omega_k} - f_k)_k$. The lower line is a canonical resolution (see 1.3.). To show exactness in the upper line we have to prove surjectivity of q. We even give a right inverse. Therefore we choose $\varphi_k \in \mathcal{D}(\Omega_k)$, $\varphi_k \equiv 1$ on Ω_{k-1} ($\Omega_0 := \emptyset$) and put

$$R\ ((f_k)_k) = (\sum_{\nu=1}^{k} \varphi_\nu f_\nu - f_k)_k \quad .$$

This defines a continuous, linear map $R : \prod_k C^\infty(\Omega_k) \to \prod_k C^\infty(\Omega_k)$. Since

$$\sum_{\nu=1}^{k+1} \varphi_\nu f_\nu - f_{k+1} - \sum_{\nu=1}^{k} \varphi_\nu f_\nu + f_k = f_{k+1} - f_{k+1} + f_k = f_k$$

on Ω_k we have $q \circ R = \mathrm{id}$.

Application of (I) gives the following commutative diagram with exact lines

$$\begin{array}{ccccccc} \dots \to \prod_k L(E,C^\infty(\Omega_k)) & \xrightarrow{q^*} & \prod_k L(E,C^\infty(\Omega_k)) & \xrightarrow{\delta^0} & \mathrm{Ext}^1(E,C^\infty(\Omega)) & \to \dots \\ & & \uparrow j^* & & \uparrow \iota^1 & \\ & \dots \to & \prod_k L(E,H_k) & \xrightarrow{\delta^0} & \mathrm{Ext}^1(E,\mathcal{N}(\Omega)) & \to 0 \end{array} .$$

q^* in the upper line is surjective since q has a right inverse there. Hence $\delta^0 = 0$ in the upper line and therefore $\iota^1 \circ \delta^0 = 0$ where δ^0 (lower line) is surjective. This proves the result.

If P(D) has a right inverse, then clearly $P(D)^*$ is surjective for every F. Hence $\mathrm{Ext}^1(F,\mathcal{N}(\Omega)) = 0$ for every F, in particular for $F = \omega := \mathbb{C}^{\mathbb{N}}$.

3.2. Lemma: *If* H *is a Fréchet space such that* $\mathrm{Ext}^1(\omega,H) = 0$ *then* $H/\ker \|\ \|$ *is a Banach space for every continuous seminorm* $\|\ \|$ *on* H.

Proof: Since the assumption implies $\mathrm{Ext}^1(\omega,H/L) = 0$ for any closed subspace $L \subset H$ it suffices to show: If H is a Frêchet space with continuous norm such that $\mathrm{Ext}^1(\omega,H) = 0$ then H is a Banach space.

We may assume that $\|\ \|_1 \le \|\ \|_2 \le \dots$ is a fundamental system of continuous norms on H. We call H_k the completion of the normed space $(H,\|\ \|_k)$ and $\rho_{k,1} : H_k \to H_1$ (for $k \ge 1$) the canonical extension of the identity. Then we have the canonical resolution

$$0 \to H \to \prod_k H_k \xrightarrow{q} \prod_k H_k \to 0 \quad ,$$

where $q((x_k)_k) = (\rho_{k+1,k}\, x_{k+1} - x_k)_k$.

If H is not a Banach space then for each $k \geq 1$ we can find $a_k \in H_k$ such that $\rho_{k,1}\, a_k \notin H$. Otherwise we would have $\rho_{k,1}\, H_k = H$ for some k which, by the closed graph theorem, would imply that H is isomorphic to the Banach space $H_k/\ker \rho_{k,1}$.

For $\xi = (\xi_1, \xi_2, \ldots) \in \omega$ we put $A(\xi) = (\xi_1 a_1, \xi_2 a_2, \ldots) \in \prod_k H_k$. Then by assumption for $A \in L(\omega, \prod_k H_k)$ there exists $B \in L(\omega, \prod_k H_k)$ such that $A = q \circ B$. We put $B e_j = (b_{1,j}, b_{2,j}, \ldots)$ where $e_j = (\delta_{k,j})_k$ and obtain:

(1) $\rho_{k+1,k}\, b_{k+1,j} = b_{k,j}$ for $k > j$ and $k < j$

(2) $a_j = \rho_{j+1,j}\, b_{j+1,j} - b_{j,j}$

(3) For each k we have j_k such that $b_{k,j} = 0$ for $j \geq j_k$.

(1) implies that there is $b_j \in H$ such that $b_{k,j} = b_j$ for all $k \geq j$. If we choose $j \geq j_1$, we have because of (2), the second part of (1) and (3)

$$\rho_{j,1}\, a_j = b_j - \rho_{j,1}\, b_{j,j} = b_j - b_{1,j} = b_j \in H$$

which contradicts the choice of a_j.

We return to our hypoelliptic operator P(D). If it has a right inverse in $C^\infty(\Omega)$ we know that $\mathrm{Ext}^1(\omega, \mathcal{N}(\Omega)) = 0$. We choose a compact $K \subset \Omega$ with non empty interior and apply Lemma 2 to the seminorm $\|f\| = \sup_{t \in K} |f(t)|$. Then $\mathcal{N}(\Omega)/\ker \|\ \| \cong \mathcal{N}(\Omega)_{|K}$ is a nuclear Banach space, hence finite dimensional. This is possible only for n = 1. Therefore we proved the main result of this section:

<u>3.3. Theorem</u>: *If* $n \geq 2$ *and* P(D) *is hypoelliptic then* P(D) *has no right inverse in* $C^\infty(\Omega)$.

For the elliptic case this result was first proved by Grothendieck (s. [21], Appendix C or [22], p. 4o f). Theorem 3 implies that also parabolic operators (the heat equation e.g.) have no right inverses. For hyperbolic operators it is known that they have right inverses at least in $C^\infty(\mathbb{R}^n)$.

4. In Grothendieck [8], II § 4 and also in [27] relations between Köthe matrices resp. Frêchet spaces are investigated which lead in concrete examples to results very similar to the relation $\mathrm{Ext}^1(E,F) = 0$.

We explain, up to some extent, the connection. In particular we show how the results on $\mathrm{Ext}^1(E,F) = 0$ can be used to solve a problem of Grothendieck in [8], II, § 4, p. 118 (s. Thm. 4.6. below) and to complete his investigation of the relation $E_b' \hat{\otimes}_\pi F$ in the case that one of the spaces is a power series space satisfying a certain stability condition. In fact we get a rather complete answer. A more systematic approach can be found in [33]. The crucial lemma 4.4. is a specialized version of a result proved there. For spaces with basis see [13].

We assume $E = \lambda^\infty(A)$, $F = \lambda(B)$, E a Schwartz space and put $P = E_b' \hat{\otimes}_\pi F$. P can be considered as a space of double indexed sequences or of matrices. The dual P' can be identified with the space of all matrices belonging to bounded linear maps from F to E. By P^* we denote the Köthe dual of P, i.e. the space of all matrices $v = (v_{i,j})_{i,j \in \mathbb{N}}$ such that

$$\sum_{i,j} |u_{i,j} v_{i,j}| < +\infty$$

for all $u = (u_{i,j})_{i,j \in \mathbb{N}} \in P$. E_+ means the set of nonnegative real sequences in $E = \lambda(A)$.

<u>4.1. Theorem</u> (Grothendieck): *The following are equivalent:*

(a) P *is bornological*

(b) P *is barrelled*

(c) $P' = P^*$

(d) $\forall n_0 \; \exists m_0 \; \forall m, \; \lambda \in E_+ \; \exists R > 0 \; \forall i,j$:

$$\frac{a_{i,m}}{b_{j,m_0}} \leq R \max\left(\frac{1}{\lambda_i \, b_{j,m}}, \frac{a_{i,m_0}}{b_{j,n_0}}\right) .$$

It is easy to see that we can write down (S_2^*) in the following way:

(S_2^*) $\forall n_0 \; \exists m_0 \; \forall m \; \exists n, \; R > 0 \; \forall i,j$:

$$\frac{a_{i,m}}{b_{j,m_0}} \leq R \max\left(\frac{a_{i,n}}{b_{j,m}}, \frac{a_{i,m_0}}{b_{j,n_0}}\right) .$$

Both conditions on the matrices are obviously related. We have even (see [13]):

<u>4.2. Proposition</u>: *Conditions* 4.1.*(d) and* (S_2^*) *are equivalent.*

At least for one part of 4.2. we shall now give a proof not involving any special assumptions on E or F, in particular they are not assumed to be Köthe spaces.

We recall that for any complete locally convex space X the following implications hold:

$$X \text{ bornological} \Rightarrow X \text{ barrelled} \Rightarrow X_b' \text{ sequentially complete.}$$

We shall show that sequential completeness of P_b', $P = L_b(E,F)$ or $P = E' \hat{\otimes}_\pi F$ implies (S_2^*).

For the proof which has some similarity with the proof of Theorem 7 in [4] we use the following notation: A projection P in F is called μ-admissible if the range of P is finite dimensional and

$$Px = \sum_j x_j\, y_j(x)$$

with $y_j \in F_\mu'$.

4.3. Lemma: (S_2^*) *is equivalent to the following condition: For every* μ *there exist* n_0, k *such that for all* K,m *there exist* n,S *and a* μ*-admissible projection* P *with*

$$\|x\|_m\, |y|_k^* \le S(\|x\|_n\, |y|_K^* + \|x\|_{n_0}\, |y|_\mu^*)$$

for all $x \in E$ *and* $y \in (\ker P)_\mu'$. $|\ |_\mu^*$ *etc. denote the dual norms in* $(\ker P)'$.

Proof: We put $Q = \mathrm{id} - P$ and assume $\mu < k < K$. With C such that $\|Qx\|_j \le C\|x\|_j$ for $j = \mu,k,K$ we have $|y|_j^* \le \|y \circ Q\|_j^* \le C\, |y|_j^*$ for these j.

We obtain

$$\|x\|_m\, \|y \circ Q\|_k^* \le C\, S\, (\|x\|_n\, \|y \circ Q\|_K^* + \|x\|_{n_0}\, \|y \circ Q\|_\mu^*)$$

for all $x \in E$, $y \in (\ker P)_\mu'$, hence for all $y \in F_\mu'$. Since P has finite dimensional range an analogous inequality with Q replaced by P is trivial.

We add these inequalities and observe that $\|y \circ Q\|_j^* + \|y \circ P\|_j^*$ is equivalent to $\|y\|_j^*$ for $j = \mu,k,K$. This proves the assertion.

4.4. Proposition: *If* P_b' *is sequentially complete for* $P=L_b(E,F)$ *or* $P=E_b' \hat{\otimes}_\pi F$, *then* E *and* F *satisfy condition* (S_2^*).

Proof: Under the assumption that (S_2^*) does not hold we construct double indexed sequences $x_{k,n}$ in E and $y_{k,n}$ in F' such that the series

$$\sum_{k,n} \langle y_{k,n}, \varphi\, x_{k,n} \rangle$$

converges uniformly on every equicontinuous set of maps $\varphi \in L(E,F)$ and such that the map $A \in L(F,E)$ defined by

$$Ax = \sum_{k,n} x_{k,n}\, y_{k,n}(x)$$

is not bounded.

Since the equicontinuous subsets of L(F,E) are the bounded sets in $L_b(E,F)$, since furthermore the canonical map $E_b' \hat{\otimes}_\pi F \to L_b(E,F)$ is continuous and under the canonical identifications $(E_b' \hat{\otimes}_\pi F)'$ corresponds to the bounded maps in L(F,E) (if their image is contained in $E \subset E''$) this proves the assertion.

We apply lemma 4.3. Then by assumption we have μ (we can assume $\mu = 1$) such that for every k (with $n_0 = k-1$) there exist K (we can assume $K = k+1$) and m (we can assume $m = k$) such that for all n and C and for all 1-admissible projections P we have $x \in E$, $y \in (\ker P)_1'$ with

$$(*)\qquad \begin{aligned} &\|x\|_k \, |y|_k^* > C\,(\|x\|_n \, |y|_{k+1}^* + \|x\|_{k-1} \, |y|_1^*) \\ &\|x\|_k = n, \quad |y|_k^* = 2\ . \end{aligned}$$

We choose a bijection $\nu \to (k(\nu), n(\nu))$ from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$ and set up an induction on ν. For $\nu = 1$ and $k = k(1)$, $n = n(1)$ we put $C = 8$, $P = 0$ and choose $x_{k,n}$, $y_{k,n}$ according to (*). We choose $a_{k,n} \in F$ with $\|a_{k,n}\|_k \le 1$, $y_{k,n}(a_{k,n}) = 1$.

Let $x_{k,n}$, $y_{k,n}$, $a_{k,n}$ be chosen for $k = k(\mu)$, $n = n(\mu)$ and $\mu = 1, \dots, \nu$ such that (*) is satisfied and $y_{k,n}(a_{l,m}) = \delta_{k,l}\,\delta_{n,m}$.

We put $P(x) = \sum_{\mu=1}^{\nu} x_{k(\mu),n(\mu)}\, y_{k(\mu),n(\mu)}(x)$. P is a 1-admissible projection. We put $k = k(\nu+1)$, $n = n(\nu+1)$, $Q = id - P$ and choose C_0 such that $\|Qx\|_j \le C_0 \|x\|_j$, for $j = 1,k,k+1$. Now we set $C = C_0\, 2^{k+n+1}$ and find $x_{k,n} \in E$, $\tilde{y}_{k,n} \in (\ker P)'$ according to (*), i.e. such that with $y_{k,n} = \tilde{y}_{k,n} \circ Q$

$$\begin{aligned} &\|x_{k,n}\|_k \, |\tilde{y}_{k,n}|_k^* > 2^{k+n+1}\,(\|x_{k,n}\|_n \, \|y_{k,n}\|_{k+1}^* + \|x_{k,n}\|_{k-1} \, \|y_{k,n}\|_1^*)\,. \\ &\|x_{k,n}\|_k = n, \quad |\tilde{y}_{k,n}|_k^* = 2\ . \end{aligned}$$

We choose $a_{k,n} \in \ker P = \operatorname{im} Q$ such that $\|a_{k,n}\|_k = 1$ and $y_{k,n}(a_{k,n}) = \tilde{y}_{k,n}(a_{k,n}) = 1$.

We obtain double indexed sequences $x_{k,n} \in E$, $y_{k,n} \in F'$, $a_{k,n} \in F$ such that:

(1) $\max\,(\|x_{k,n}\|_n \, \|y_{k,n}\|_{k+1}^*, \|x_{k,n}\|_{k-1} \, \|y_{k,n}\|_1^*) < n\, 2^{-k-n}$

(2) $\|x_{k,n}\|_k = n$, $\|a_{k,n}\|_k = 1$

(3) $y_{k,n}(a_{l,m}) = \delta_{k,l}\,\delta_{n,m}$.

Every equicontinuous subset of $L(E,F)$ is contained in one of the form $B = \{\varphi \in L(E,F) : \|\varphi x\|_k \le C(k)\ \|x\|_{n(k)}$ for all $k\}$ with increasing sequences $n(k)$ in $\mathbb{N}$ and $C(k) \ge 0$.

Given B we put $l = n(l)$, $m = n(l+1)$ and obtain for $\varphi \in B$

$$\sum_{k,n} |\langle y_{k,n}, \varphi x_{k,n}\rangle| \le C(l) \sum_{k=l+1}^{\infty} \sum_{n} \|x_{k,n}\|_{k-1} \|y_{k,n}\|_1^* +$$

$$+ C(l+1) \sum_{k=1}^{l} \sum_{n=m}^{\infty} \|x_{k,n}\|_n \|y_{k,n}\|_{k+1}^* +$$

$$+ C(l) \sum_{k=1}^{l} \sum_{k=1}^{m-1} \|x_{k,n}\|_l \|y_{k,n}\|_1^* \le$$

$$\le C(l) \sum_{k=l+1}^{\infty} \sum_{n} n\ 2^{-k-n} + C(l+1) \sum_{k=1}^{\infty} \sum_{n=m}^{\infty} n\ 2^{-k-n} + \ldots .$$

Since the estimate is termwise, the series on the left hand converges uniformly on B.

For $A \in L(F,E)$ as defined above we have $A\ a_{k,n} = x_{k,n}$ hence $\|A\ a_{k,n}\|_k = n$ while $\|a_{k,n}\|_k = 1$ for all k and n. Therefore for every k the set $\{A\ x : \|x\|_k \le 1\}$ is not bounded, i.e. A is not bounded.

Since all the necessary conditions for $\mathrm{Ext}^1(E,F) = 0$ in [31] are based on (S_2^*) they all are necessary conditions for the bornologicity or barrelledness of $L_b(E,F)$. Instead of a converse of Prop. 4.4. we complement it by a sufficient condition.

<u>4.5. Proposition</u>: *If* E *and* F *are nuclear,* E *has property* (DN) *and* F *has property* (Ω), *then* $L_b(E,F) \cong E_b' \hat{\otimes}_\pi F$ *is bornological.*

Proof: From Grothendieck's result [8], II, § 4, n°3, Cor. 2 (cf. 4.1. and 4.2.) we know that $L_b(s,s)$ is bornological.

From [24], Prop. 3.3. and 4.5., by adding complements we obtain exact sequences

$$0 \to E \xrightarrow{\iota} s \to G \to 0$$

$$0 \to H \to s \xrightarrow{q} F \to 0$$

where G and H are complemented subspaces of s.

Since $\mathrm{Ext}^1(E,H) = 0$ and $\mathrm{Ext}^1(G,s) = 0$ we obtain that the map

$$\begin{array}{ccc} L_b(s,s) & \longrightarrow & L_b(E,F) \\ \wr\| & & \wr\| \\ s_b' \hat{\otimes}_\pi s & \xrightarrow{\iota^* \otimes q} & E_b' \hat{\otimes}_\pi F \end{array}$$

is surjective. By [8], I, § 1, n^o 2, Prop. 3 $\iota^* \otimes q$ is a topological homomorphism. Hence $E_b' \hat{\otimes}_\pi F = L_b(E,F)$ is bornological.

Let V be a complex analytic manifold. In [8], II, § 4, p. 128 Grothendieck states that $L_b(\mathcal{H}(V), \mathcal{H}(V))$ is bornological if $V = \mathbb{C}$, whereas for V = D (unit disc) it is not. He asks for a classification of complex manifolds with respect to linear topological properties of $\mathcal{H}(V)$. Now for the present case we can give a complete answer.

4.6. Theorem: *The following are equivalent:*

(1) $L_b(\mathcal{H}(V), \mathcal{H}(V))$ *is bornological*

(2) $L_b(\mathcal{H}(V), \mathcal{H}(V))$ *is barrelled*

(3) $\mathcal{H}(V)$ *has property* (DN)

(4) every plurisubharmonic function on V, *which is bounded from above is constant (strong Liouville property).*

Proof: We have (1) ⇒ (2) ⇒ P_b' sequentially complete for $P = L_b(\mathcal{H}(V), \mathcal{H}(V))$. By 4.4. this implies (S_2^*) for $E = F = \mathcal{H}(V)$. Since [31], Thm. 7.2. is completely based on (S_2^*) we know from that theorem that $\mathcal{H}(V)$ has (DN).

(3) ⇒ (1) follows from 4.5. since $\mathcal{H}(V)$ is nuclear and has property (Ω) (cf. [31] § 7,B).

The equivalence of (3) and (4) is shown in [37].

Before we treat the case of power series spaces we need some information about the connection between the results of [27] and the present paper. By LB(F,E) we denote the space of all bounded linear maps from F to E, i.e. those maps which send some neighbourhood of zero into a bounded set. We obtain from [27], 1.3. and 1.4.

4.7. Theorem: *If* $E = \lambda^\infty(A)$ *or* $F = \lambda(B)$ *then the following are equivalent:*

(1) $L(F,E) = LB(F,E)$

(2) for every sequence K(N) *there exists* K *such that for each* n *we have* N_0 *and* C *with*

$$\|x\|_n \, \|y\|_K^* \leq C \max_{N=1,\dots,N_0} \|x\|_N \, \|y\|_{K(N)}^*$$

for all $x \in E$, $y \in F'$.

One part of [27], 7.3. says that at least for reflexive E the equality L(F,E) = LB(F,E) implies that $E_b' \otimes_\pi F$, hence also $E_b' \hat{\otimes}_\pi F$ is barrelled.

We recall that a Fréchet space F has property $(\bar{\Omega})$ if the following holds (s. [36], [26]):

$(\bar{\Omega})$ $\forall k\ \exists K\ \forall L\ \exists C > 0\quad \forall y \in F' : \|y\|_K^{*2} \leq C\, \|y\|_L^* \, \|y\|_k^*$.

One connection between [27] and the present paper is given by the following lemma which explains in a very satisfactory way the coincidence of the conditions in [31], 4.2. and [27], 4.3. as well as in [31], 4.3. (r=1) and [27], 2.1. at least for nuclear $\Lambda_1(\alpha)$.

<u>4.8. Lemma</u>: *If* $E = \lambda^\infty(A)$ *or* $F = \lambda(B)$ *and* F *has property* $(\bar{\Omega})$ *then* (S_1^*) *implies* L(F,E) = LB(F,E).

<u>Proof</u>: (S_1^*) yields n_o. For a given sequence K(N) we put $\mu = K(n_o)$ and obtain from (S_1^*) a number k such that for all K and m we have n and S with

$$\|x\|_m \leq S \left(\|x\|_n \frac{\|y\|_K^*}{\|y\|_k^*} + \|x\|_{n_o} \frac{\|y\|_{K(n_0)}^*}{\|y\|_k^*} \right)$$

for all $x \in E$, $y \in F'$.

For k we choose $K \geq k$ according to $(\bar{\Omega})$ and obtain for L = K(n), n = n(K,m) as in (S_1^*), a C > 0 such that

$$\frac{\|y\|_K^*}{\|y\|_k^*} \leq C \frac{\|y\|_{K(n)}^*}{\|y\|_K^*}$$

for all $y \in F'$.

We obtain for $x \in E$, $y \in F'$:

$$\|x\|_m \, \|y\|_K^* \leq (C + S) \max \left(\|x\|_n \, \|y\|_{K(n)}^*, \|x\|_{n_o} \, \|y\|_{K(n_o)}^* \right)$$

which proves the assertion.

<u>Remark</u>: A similar proof shows that (S_1^*) and (S_2^*) are equivalent if E has property (DN).

In the following theorem we assume for (3), α. and β. that $\sup_n \alpha_{n+1}/\alpha_n < +\infty$

and for (3), γ. that $\lim_n \alpha_{n+1} / \alpha_n = 1$ *(stability assumptions)*.

A Fréchet space F has property $(\overline{\overline{\Omega}})$ if the following holds (s. [27]; [31]) :

$(\overline{\overline{\Omega}})$ $\forall k, \varepsilon > 0 \ \exists K \ \forall L \ \exists C > 0 \ \forall y \in F' : \|y\|_K^{*\,1+\varepsilon} \le C \|y\|_L^* \|y\|_k^{*\,\varepsilon}$.

Obviously property $(\overline{\overline{\Omega}})$ implies $(\overline{\Omega})$.

The theorem extends (in the nuclear case) Prop. 15 in [8], II, §4, n°3 which gives the case α. for E a Köthe space.

4.9. Theorem: *If* E *and* F *are nuclear and one of them a power series space satisfying our stability assumptions, then the following are equivalent:*

(1) $L_b(E,F)$ *is bornological*

(2) $L_b(E,F)$ *is barrelled*

(3) α. *if* $F = \Lambda_r(\alpha)$: E *has property* (DN)

β. *if* $E = \Lambda_\infty(\alpha)$: F *has property* (Ω)

γ. *if* $E = \Lambda_1(\alpha)$: F *has property* $(\overline{\overline{\Omega}})$.

Proof: As in 4.6. we have (1) ⇒ (2) ⇒ (S_2^*). Since the necessity parts of thms. 4.1., 4.2., 4.3. in [31] are based on (S_2^*), these theorems yield (2) ⇒ (3). In the cases α. and β. (3) ⇒ (1) follows from 4.5., in the case γ. condition (S_1^*) is satisfied as we know from [3], 4.2. the sufficiency part of which is based on (S_1^*). 4.8. yields L(F,E) = LB(F,E), which implies barrelledness of $E_b' \hat{\otimes}_\pi F \cong L_b(E,F)$ (s. [27], 7.3.). Finally Theorem 14.3. b in [8], II, § 4 n° 2 tells that in this case (2) ⇒ (1).

5. We want to determine those Fréchet spaces E and F which can occur in a non-trivial way in the relation $Ext^1(E,F) = 0$. We shall use the theory of [31]. Hence the main tool is an investigation of the relations (S_1^*) and (S_2^*).

For a given fundamental system of seminorms in F we denote by F_k the canonical Banach spaces. The dual of F_k can be identified with the space F_k', its norm with $\| \ \|_k^*$. We use the following condition introduced by Bellenot and Dubinsky [3] to characterize the Fréchet spaces which have nuclear Köthe quotients with continuous norm

$$(*) \qquad \exists \mu \ \forall k \ \exists K : \sup \{\|y\|_k^* : y \in F_\mu', \ \|y\|_K^* \le 1\} = +\infty.$$

Typical examples for spaces not satisfying (*) are products of Banach spaces or, more general, quojections (s. [3]). The only Fréchet Montel space which does not satisfy (*) is ω.

5.1. Proposition: *If* F *does not satisfy* (*) *then* (S_1^*) *is satisfied for every Fréchet space* E.

The proof is obvious.

With respect to the following analysis (Lemma 5.3. and Thm. 5.5.) the proposition means that for F not satisfying (*) we cannot get any information on E from (S_1^*) or from (S_2^*). However it is only known under additional assumptions that (S_1^*) implies $\mathrm{Ext}^1(E,F) = 0$ (see e.g. Thm. 1.9. or [31], § 3).

If we restrict our attention mainly to the cases where either E or F is contained in the class of nuclear spaces then the case where F is nuclear causes no difficulty since in this case (S_1^*) implies $\mathrm{Ext}^1(E,F) = 0$.

In the other case we can give a precise characterization of the spaces F for which $\mathrm{Ext}^1(E,F) = 0$ for every E. Following Bellenot and Dubinsky [3] we call quojections the Fréchet spaces F where $F/\ker \| \ \|$ is a Banach space for any continuous seminorm $\| \ \|$. These are the spaces occuring in Lemma 3.2.. From Lemma 3.2. and an argument in [31], we obtain:

5.2. Theorem: *The following are equivalent for a Fréchet space* F*:*

(1) F *is a quojection*
(2) $\mathrm{Ext}^1(\omega,F) = 0$
(3) $\mathrm{Ext}^1(E,F) = 0$ *for every nuclear Fréchet space* E.

From [3] we know the following. If F is a quojection then it does not satisfy (*). The converse is true for reflexive F. It is unknown if it is true always, i.e. if F is a quojection iff it does not satisfy (*). Hence we do not know if (for E restricted to the class of nuclear spaces) 5.1. and 5.5. cover everything. But 5.1. says that 5.5. (1) is optimal as long as we use conditions (S_1^*) and (S_2^*) for our analysis.

While in the preceding we discussed cases where we cannot get any information on E from (S_1^*) resp. (S_2^*) or from $\mathrm{Ext}^1(E,F) = 0$, our main objective is now to investigate the case of spaces with (*). We need the following generalization of (DN) (cf. [2], [34]). Let φ always denote a strictly increasing continuous function $(0,+\infty) \to (0,+\infty)$, $\lim_{r\to\infty} \varphi(r) = +\infty$.

$$(DN_\varphi) \quad \exists n_0 \ \forall m \ \exists n, C \quad \forall x \in E,\ r > 0 :$$
$$\|x\|_m \le C\varphi(r)\, \|x\|_{n_0} + \frac{1}{r}\, \|x\|_n .$$

5.3. Lemma: *If* F *satisfies* (*) *and* E *and* F *satisfy* (S_2^*) *then* E *has property* (DN_φ) *for some* φ.

Proof: We choose μ according to (*) and obtain from (S_1^*) n_o and k. (*) again yields K and putting this into (S_2^*) we finally obtain: For every m there exists n and S such that for all $x \in E$ and $y \neq 0 \in F'_\mu$ we have:

$$\|x\|_m \le S \frac{\|y\|_\mu^*}{\|y\|_k^*} \|x\|_{n_o} + S \frac{\|y\|_K^*}{\|y\|_k^*} \|x\|_n .$$

We choose a sequence $y_\nu \in F'_\mu$ such that

$$\frac{\|y_\nu\|_k^*}{\|y_\nu\|_K^*} < \frac{\|y_{\nu+1}\|_k^*}{\|y_{\nu+1}\|_K^*} \xrightarrow[(\nu\to\infty)]{} +\infty$$

and a function φ such that for all $S > 0$ we have $C > 0$ with

$$S \frac{\|y_{\nu+1}\|_\mu^*}{\|y_{\nu+1}\|_k^*} \le C\varphi\left(\frac{1}{S} \frac{\|y_\nu\|_k^*}{\|y_\nu\|_K^*}\right)$$

for all $\nu \in \mathbb{N}$. For

$$\frac{\|y_\nu\|_k^*}{\|y_\nu\|_K^*} \le S\,r \le \frac{\|y_{\nu+1}\|_k^*}{\|y_{\nu+1}\|_K^*}$$

we obtain

$$\|x\|_m \le C\varphi(r)\, \|x\|_{n_o} + \frac{1}{r} \|x\|_n .$$

By increasing C if necessary we have this inequality for all $r > 0$ which proves (DN_φ) .

5.4. Lemma: *If* E *has property* (DN_φ) *for some* φ *then there exists a nuclear Köthe space* $\lambda(B)$ *with continuous norm such that* (S_1^*) *is satisfied.*

Proof: We choose a strictly increasing function $\psi : (0,+\infty) \to (0,+\infty)$, $\psi(r) \le r$, $\psi(1) = 1$, $\lim_{r\to+\infty} \psi(r) = +\infty$ such that

$$\lim_{r\to\infty} \frac{\psi^\nu(r)}{\varphi^{-1}(r)} = 0$$

for all $\nu \in \mathbb{N}$. Moreover we choose a sequence $\beta_j \ge 1$ such that $\sum_j \beta_j^{-1} < +\infty$.

We put

$$b_{j,1} = 1 \quad \text{for all } j$$

$$b_{j,2} = \underbrace{\psi^{-1} \circ \ldots \circ \psi^{-1}}_{(j-1)\text{times}} (\beta_j) \text{ for all } j$$

and define $b_{j,k}$ for $k > 2$ inductively by

$$b_{j,k+1} = b_{j,k} \quad \psi\left(\frac{b_{j,k}}{b_{j,k-1}}\right) \quad .$$

We obtain for $j \geq k$

$$\frac{b_{j,k+1}}{b_{j,k}} = \underbrace{\psi \circ \ \ldots \circ \psi}_{(k-1)\text{times}} (b_{j,2}) = \underbrace{\psi^{-1} \circ \ \ldots \circ \psi^{-1}}_{(j-k)\text{times}} (\beta_j)$$

$$\geq \beta_j \ .$$

Hence $\lambda(B)$ is nuclear. It has continuous norm.

To establish (S_1^*) we choose n_o such that for all m we have n,C with

(1) $$\|x\|_m \leq C\varphi(r) \|x\|_{n_o} + \frac{1}{r} \|x\|_n$$

for all $r > 0$ and $x \in E$. For given μ we put $k = \mu + 1$ and obtain for all $K > k$ and large j

$$\frac{b_{j,K}}{b_{j,k}} = \prod_{\nu=\mu}^{K-1} \frac{b_{j,\nu+1}}{b_{j,\nu}} \leq \psi^{K-\mu} \left(\frac{b_{j,k}}{b_{j,\mu}}\right)$$

$$\leq \varphi^{-1} \left(\frac{b_{j,k}}{b_{j,\mu}}\right) \quad .$$

In (1) we put $r = \dfrac{b_{j,K}}{b_{j,k}}$ and obtain for large j

$$\|x\|_m \leq C \frac{b_{j,k}}{b_{j,\mu}} \|x\|_{n_o} + \frac{b_{j,k}}{b_{j,K}} \|x\|_n$$

hence

$$\frac{\|x\|_m}{b_{j,k}} \leq S \quad \frac{\|x\|_n}{b_{j,K}} + \frac{\|x\|_{n_o}}{b_{j,\mu}}$$

for all j with appropriate $S > 0$, which proves the assertion (cf. 4.2.).

Lemma 5.3. and 5.4. together give the following theorems. Proposition 5.1. tells us that the restriction on F is natural.

5.5. Theorem: *The following are equivalent:*

(1) there exists F *satisfying* (*) *such that* $\mathrm{Ext}^1(E,F) = 0$

(2) there exists a nuclear F *not isomorphic to* ω *such that* $\mathrm{Ext}^1(E,F) = 0$

(3) E *has property* (DN_φ) *for some* φ.

Remark: A trivial consequence is that (1) or (2) implies that E admits a continuous norm.

To give a more careful analysis of the meaning of (3) in 5.5. we use the following condition (s. [28]):

(**) There exists p_0 such that for every $p \geq p_0$ we have a $q \geq p$ with the following property: every sequence in E which is Cauchy w.r.t. $\| \ \|_q$ and converges to 0 w.r.t. $\| \ \|_{p_0}$ even converges to 0 w.r.t. $\| \ \|_p$.

This condition is equivalent to E being countably normable (s. [7]), i.e. to the existence of a fundamental system of norms on E which makes it into a countably normed space in the sense of Gelfand - Šilov.

5.6. Lemma: *The following are equivalent :*

(1) E *has property* (DN_φ) *for some* φ

(2) the analogue to (**) *with "Cauchy" replaced by "bounded" holds.*

Proof: It is easy to prove that (1) implies (2). We assume (2) and show that the property there implies:

$$\|x\|_p \leq \varphi_{p,p_0,q}(r) \ \|x\|_{p_0} + \frac{1}{r} \ \|x\|_q$$

for all $x \in E$, $r > 0$ and an appropriate function $\varphi(r) = \varphi_{p,p_0,q}(r)$.

For this it suffices to show that for every $r > 0$ there exists M such that for all $x \in F$

$$\|x\|_p \leq M \ \|x\|_{p_0} + \frac{1}{r} \ \|x\|_q \quad .$$

We assume that this were not true. Then there exists a $r > 0$ such that for every M we have $x_M \in E$ with

$$\|x_M\|_p > M\,\|x_M\|_{p_0} + \frac{1}{r}\,\|x_M\|_q \quad .$$

In particular $\|x_M\|_p > 0$. Hence we can assume $\|x_M\|_p = 1$. But then $\|x_M\|_{p_0} \to 0$ and $\|x_m\|_q \leq r$ for all M which contradicts (2).

We choose φ such that

$$\lim_{r\to\infty} \frac{\varphi_{p,p_0,q}(r)}{\varphi(r)} = 0$$

for all p,p_0,q which occur. It follows that E has property (DN_φ).

If E is a Schwartz space then for every p we have $\tilde{p}$ such that every sequence which is bounded w.r.t. $\tilde{p}$ has a subsequence which is Cauchy w.r.t. p. This proves the second part of the following lemma, the first one is obvious from 5.6. .

<u>5.7. Lemma</u>: *(1) If* E *has property* (DN_φ) *for some* φ *then it is countably normable.*
(2) If E *is a countably normable Schwartz space then it has property* (DN_φ) *for some* φ .

Hence theorem 5.5. gives for the case of Schwartz spaces:

<u>5.8. Theorem</u>: *For a Schwartz space* E *the following are equivalent:*
(1) there exists a Schwartz space (nuclear space) F *not isomorphic to* ω *such that* $\mathrm{Ext}^1(E,F) = 0$
(2) E *is countably normable.*

In Dubinsky [5], [6] and in [7], [28] there are given examples of nuclear (F)-spaces with continuous norm but not countably normable which therefore do not have the bounded approximation property. From the preceding we conclude that there are also examples of nuclear spaces E with continuous norm for which there is no F satisfying (*) such that $\mathrm{Ext}^1(E,F) = 0$. For F satisfying (*) and nuclear E 5.1. implies $\mathrm{Ext}^1(E,F) = 0$.

To determine conditions on F for the existence of E such that $\mathrm{Ext}^1(E,F) = 0$ we recall the following concept which was introduced by Grothendieck [9] : A locally convex space F is called quasi-normable if for every equicontinuous set $A \subset F'$ there is a neighbourhood of zero V in F such that on A the topology of F'_b coincides with the topology of uniform convergence on V.

Equivalently (s. [9], p. 1o7): F is quasi-normable if and only if for every neighbourhood of zero U there exists V such that for each $\alpha > 0$ there exists a bounded set $B \subset F$ such that

$$V \subset B \quad + \alpha U .$$

In [14] it is shown that a Fréchet space F is quasinormable if and only if there exists φ such that F has the property

$$(\Omega_\varphi) \quad \forall \mu \; \exists k \; \forall K \; \exists C \; \forall r > 0, y \in F' : \\ \|y\|_k^* \leq C\,\varphi(r)\, \|y\|_K^* + \frac{1}{r} \|y\|_\mu^* .$$

5.9. Lemma: *If* E *and* F *satisfy* (S_2^*) *and* E *is a proper (non normable) Fréchet space, then* F *has property* (Ω_φ) *for some* φ .

Proof: If F does not satisfy condition (*) then it obviously has property (Ω_φ) for every φ and hence it is quasi-normable.

If it satisfies condition (*) then we know from the remark after 5.5. that E admits a continuous norm. We can assume that all $\| \; \|_k$ on E are norms. In an analogous way as in the proof of 5.3. we show that F has property (Ω_φ) for some φ .

Given μ we find n_0 and k according to (S_2^*). We choose $m > n_0$ such that there exists a sequence $x_\nu \in E$ such that

$$\frac{\|x_\nu\|_m}{\|x_\nu\|_{n_0}} < \frac{\|x_{\nu+1}\|_m}{\|x_{\nu+1}\|_{n_0}} \xrightarrow[(\nu\to\infty)]{} + \infty \quad .$$

This is possible since F is not normable. For given K and m we obtain n and S such that for all $y \in F'$ and ν :

$$\|y\|_k^* \leq S \frac{\|x_\nu\|_n}{\|x_\nu\|_m} \|y\|_K^* + S \frac{\|x_\nu\|_{n_0}}{\|x_\nu\|_m} \|y\|_\mu^* .$$

We continue as in the proof of 5.3. .

The existence of a nuclear $\lambda(A)$, even with some additional properties, for given F with property (Ω_φ), such that $\mathrm{Ext}^1(\lambda(A),F) = 0$ is implicitly shown in [14]. But since we do not need here the additional properties we give a direct proof.

5.1o. Lemma: *If* F *has property* (Ω_φ) *for some* φ *then there exists a nuclear Köthe space* $E = \lambda(A)$ *such that* (S_1^*) *is satisfied.*

Proof: We can assume $\varphi(r) \geq r$, $\varphi(1) = 1$. We choose $\alpha_j \geq 1$ such that $\sum_j \alpha_j^{-1} < + \infty$

and put

$$a_{j,1} = 1 \quad \text{for all } j$$

$$a_{j,2} = \alpha_j \quad \text{for all } j \quad .$$

For $k > 2$ we define $a_{j,k}$ inductively by

$$a_{j,k+1} = a_{j,k} \quad \varphi(a_{j,k}).$$

An induction argument shows that $\varphi(a_{j,k}) \geq \alpha_j$ for $k \geq 2$ hence

$$\frac{a_{j,k+1}}{a_{j,k}} = \varphi(a_{j,k}) \geq \alpha_j$$

which implies that $\lambda(A)$ is nuclear.

The proof continues as the proof of 5.4.

The equivalence of (2) and (3) in the following theorem is contained in [14], theorem 6. It should be noticed that here we do not need the more sophisticated construction of $\lambda(A)$ given in [14]. The equivalence of (1) and (2) is contained in Lemmas 5.9. and 5.1o.

5.11. Theorem: *For a Fréchet space* F *the following are equivalent:*

(1) there exists a proper Fréchet space E *such that* $\mathrm{Ext}^1(E,F) = 0$

(2) F *has property* (Ω_φ) *for some* φ

(3) F *is quasi-normable* .

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